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Closure Properties of Order Continuous Operators

Introduction

Let X be a compact Hausdorff space and let C(X) (or simply C) be the space of all real valued continuous functions on X. C'(X) and C"(X) (or C', C" respectively) represent the first and second norm duals of C(X). In [4], Kaplan studied the order closure of C when imbedded in C". We will consider an analogous question about operators by imbedding the space of operators from C to itself in the space of order continuous operators from C" to C".

1. Preliminaries

C, C', and C" are examples of *Riesz spaces* (or *vector lattices*), ordered vector spaces where the supremum and infimum of two elements exist.

If E is a Riesz space and $\{x_\alpha\}$ is an increasing (decreasing) net in E, we say that x_α *order converges* to x∈E if $x = \vee_\alpha x_\alpha (x = \wedge_\alpha x_\alpha)$, and we write $x_\alpha \uparrow x (x_\alpha \downarrow x)$. More generally, we say that net $\{x_\alpha\}$ which is not necessarily monotone order converges to x if there are nets $\{y_\alpha\}$ and $\{z_\alpha\}$ such that $y_\alpha \downarrow x, z_\alpha \uparrow x$ and $z_\alpha \leqq x_\alpha \leqq y_\alpha$. We write $x_\alpha \rightarrow x$ or $x = lim_\alpha x_\alpha$. Unless otherwise specified, any reference to limits, convergence, denseness, etc. will be in the sense of order convergence. A Riesz space is *Dedekind complete* if every set which is bounded above has a supremum. If x_α is a bounded net in a Dedekind complete Riesz space, then the following are always defined.

$$limsup_\alpha x_\alpha = \wedge_\alpha \vee_{\beta \geq \alpha} x_\beta$$

$$liminf_\alpha x_\alpha = \vee_\alpha \wedge_{\beta \geq \alpha} x_\beta$$

A subspace F $\subset$ E which is closed under finite infima and suprema is called a *Riesz subspace*. If $\{y \in E; 0 \leqq y \leqq x, x \in F\}$ is also contained in F, then F is said to be an *ideal* of E. An ideal which is closed under order convergence is called a *band*. If A is any subset of E, A^d is defined by

$$A^d = \{y \in E; \mid y \mid \wedge \mid x \mid = 0, \ all\ x \in A\}.$$

A^d is a band in E. If E is Dedekind complete and F $\subset$ E is a band, then E may be written as the direct sum of F and F^d, $E = F \oplus F^d$.

If F is an ideal of E, the positive cone of the band generated by F is obtained by taking all suprema of increasing nets in F_+.

Suppose E is Dedekind complete and F $\subset$ E is a Riesz subspace. If $x = \wedge_{y \in A} y = \vee_{z \in B} z$ for A, B $\subset$ F implies that x$\in$F, then F is said to be *Dedekind closed*.

C may be imbedded in C" in a natural way. In general, we will not distinguish between f$\in$C and the corresponding f$\in$C". If a Riesz space is also a Banach space and the norm is compatible with the order structure, i.e. $\mid x \mid \leqq \mid y \mid$ implies $\| x \| \leqq \| y \|$, then it is called a *Banach lattice*. C and C" are AM spaces, Banach lattices whose norms satisfy $\| f \vee g \| = \| f \| \vee \| g \|$ for f and g positive. Let **1** be the unit in C, the constant one function on X. **1** is also a unit for C", and by a theorem of Kakutani C" may be represented as C(Y) for some compact Hausdorff space Y. Since C" is Dedekind complete, Y is Stonian, i.e. the closure of every open set is open [6, p. 108]. We will apply this notation throughout, letting Y be the Stone space of C"(X).

An element e∈C"$_+$ will be called a *component of* **1** (or simply a *component*) if e∧(**1**-e)=0. The set of all components will be denoted by $\mathcal{E}$. Each component corresponds to an open and closed subset of Y [3, 17.4 and 31.6]. A set P⊂ $\mathcal{E}$ will be called a partition of **1** if $\vee_{e\in P} e = \mathbf{1}$ and $e_1 \wedge e_2 = 0$ for $e_1 e_2 \in$P.

For μ ∈C' and f∈C", by fμ we will mean that element of C' defined by

$$\langle f\mu, g\rangle = \langle \mu, fg\rangle,\ g \in C.$$

We will be especially interested in several subsets and subspaces of C". The following definitions and results are due to Kaplan [3].

Every element of C" which is the supremum (infimum) of a subset of C will be called lower semicontinuous (upper semicontinuous). The set of all such suprema will be denoted by lsc (usc). The Riesz subspace lsc-lsc = {f-g; f,g∈lsc} (=usc-usc) will be denoted by SC. We note that the lsc elements of C" are exactly those which are σ(C',C) (the weak-* or vague topology on C') lower semicontinuous on the positive part of the unit ball of C', and are thus also lower semicontinuous on the natural image of X in C'. In fact, each open subset of X corresponds to an element of $\mathcal{E}$∩lsc. If f∈usc and g∈lsc, f≤g, there is an h∈C with f≤h≤g.

Every f∈C" which is the limit of a net in C will be called *universally integrable*. The set of all such elements is a Riesz subspace and will be denoted by U. Both U and C are Dedekind closed in C". U is in fact the set of elements of C" which are simultaneously infima of subsets of lsc and suprema of subsets of usc.

The smallest σ-closed (closed under order convergence of sequences) subspace of C" which contains SC will be denoted by Bo. (It is possible to identify Bo with the space of Borel functions on X [3, 54.3 and 54.11]). As U is σ-closed, Bo $\subset$ U. Every f$\in$C" is the order limit of a net in Bo (or U). Both Bo and U are norm closed in C".

For $\mu \in$C', C'$_\mu$ will represent the band in C' generated by μ. C"$_\mu$ will be the band in C" dual to C'$_\mu$, i.e. if $C'^{\perp}_{\mu} = \{f \in C"; \langle |\mu|, |f| \rangle = 0\}$, then $C"_\mu = (C'^{\perp}_{\mu})^d$. For f$\in$C", f_μ will be the image of f under the projection on C"$_\mu$. C'$_\mu$ is isomorphic with the space $L^1(\mu)$, thus C"$_\mu$ may be identified with $L^\infty(\mu)$. The spaces Bo and U project *onto* C"$_\mu$. If $\{f_\alpha\} \subset C"_\mu$ and $f_\alpha \downarrow 0$, then there is a sequence $\{f_n\} \subset \{f_\alpha\}$ such that $f_n \downarrow 0$. C"$_\mu$ is an AM-space with unit $\mathbf{1}_\mu$.

For $\mu \in$C', the ideal generated by $(C"_\mu)^d \cap U$ in C" will be denoted by N_μ. N_μ and U+N_μ are σ-closed. (U+N_μ corresponds to the set of functions integrable with respect to μ.) Every element of U differs from an element of Bo by an element of $(C"_\mu)^d$; thus Bo + N_μ = U+N_μ.

If E and F are Riesz spaces, the set of all linear operators from E to F which map intervals into order bounded sets is denoted by L^b(E,F). L^b(E,F) is ordered by $T \leqq S$ when $Ty \leqq Sy$ for $y \in E_+$, but it is not necessarily a Riesz space. The subspace consisting of all differences of positive operators is called the space of regular operators, L^r(E,F). If F is Dedekind complete, L^b(E,F) is a Dedekind complete Riesz space, and for T, S $\in L^b$(E,F) and $x \in E_+$, $T \vee S$ is given by

$$T \vee Sx = \bigvee_{\substack{x_1+x_2=x \\ x_1,x_2 \in E_+}} (Tx_1 + Sx_2).$$

In this case, we have $L^r(E,F) = L^b(E,F)$. Also, the band of $L^b(E,F)$ consisting of operators which are continuous with respect to order convergence is designated by $L^c(E,F)$ and is called the space of *order continuous operators*. For $T \in L^c(C'',C'')$, T_μ is the projection onto C''_μ composed with T.

$L^b(C,C)$ may be imbedded in $L^c(C'',C'')$ by identifying each $T \in L^b(C,C)$ with its bitranspose $T^{tt} \in L^c(C'',C'')$. In general, we will not distinguish between T and T^{tt} and we will consider T as an element of $L^c(C'',C'')$ when it is convenient to do so. If $T \in L^c(C'',C'')$ and $f \in C''$, we will denote by fT the operator defined by

$$fTg = f(Tg), \quad g \in C''.$$

Because C is order dense in C'', every operator in $L^c(C'',C'')$ is determined by its values on C, and conversely every bounded operator from C to C'' may be (uniquely) extended to an order continuous operator from C'' to C''. Thus, we will use the symbol $L^r(C,U)$ to represent the subspace of $L^c(C'',C'')$ which consists of differences of positive operators mapping C to U.

For more complete information about Riesz spaces and operators, see Vulikh [7] or Schaeffer [6].

It is possible to translate the lifting theorem of Tulcea and Tulcea [1] to C'' by replacing $L^\infty(\mu)$ with C''_μ and the space of measurable functions with $Bo + N_\mu$ in the proof to obtain [2, Theorem A.1]:

1.1 Theorem (Tulcea) There exists a positive bounded linear mapping I: $C''_\mu \rightarrow C''$ which satisfies:

1. $I\mathbf{1}_\mu = \mathbf{1}$.
2. I maps $\mathcal{E} \cap C''_\mu$ into $\mathcal{E}$.
3. $(If)_\mu = f$ for all $f \in C''_\mu$.
4. I takes values in Bo + $N_\mu (=U+N_\mu)$.

We will often require two copies of $C''(X) = C(Y)$ and will denote the second by $\overline{C}''$ $= C(\overline{Y})$. We will extend this notation with $\overline{f} \in \overline{C}''$, $\overline{y} \in \overline{Y}$ and $\overline{e}$ a component in $\overline{C}'' = C(\overline{Y})$.

Each $e \in \mathcal{E}$ determines a set V(e) which is open and closed in Y. The set of all such V(e) is a basis for the topology on Y.

If $T \in L^b(C'',C'')_+$, $\mu \in C'_+$, and $f \in C''_+$, then

$$m(V(e), V(\overline{e})) = \langle \overline{e}\mu, Tfe \rangle,\ e, \overline{e} \in \mathcal{E}$$

defines a measure on $Y \otimes \overline{Y}$. If Φ is a function defined on $Y \times \overline{Y}$, we will denote the integral of Φ with respect to this measure (when it exists) by

$$\int \Phi(y, \overline{y}) \langle d\overline{e}\mu, Tfde \rangle.$$

The following is essentially due to Nakano [5, Theorem 4.3].

<u>1.2 Proposition.</u> If S, $T \in L^c(C'',C'')$ with $0 \leq T \leq S$ and $\mu \in C'_+$, then there is a Borel measurable function Φ defined on $Y \times \overline{Y}$ such that $\langle \nu, Tf \rangle = \int \Phi(y,\overline{y})\ \langle d\overline{e}\nu, Sfde \rangle$ for all $f \in C''$, and $\nu \in C'_\mu$.

2. The order closure of $L^r(C,C)$ and $L^r(C,U)$.

We begin with an important topology on $L^c(C",C")$.

2.1 Theorem. Let $\mu \in C'_+$ and $E = C$ or U. $L^r(C,E)$ is dense in the band which it generates in $L^c(C",C")$ in the topology defined by the semi-norm

$$\| T \|_\mu = \langle \mu, | T | \mathbf{1} \rangle.$$

Proof. We will first suppose that $T \in L^c(C",C")$ satisfies $0 \leqq T \leqq S$ for some $S \in L^r(C,E)$. By 1.2, there is a Borel function Φ such that $\langle \nu, Tf \rangle = \int \Phi(y,\bar{y}) \langle d\bar{e}\nu, Sfde \rangle$ holds for all $f \in C"$ and $\nu \in C'_\mu$. We may assume $0 \leqq \Phi \leqq \mathbf{1}$ $(Y \times \overline{Y})$. Given $\epsilon > 0$, there is a function Ψ which is continuous on $Y \times \overline{Y}$ such that $0 \leqq \Psi \leqq \mathbf{1}$ $(Y \times \overline{Y})$ and

$$\int | \Psi - \Phi | \langle d\bar{e}\mu, S\mathbf{1}de \rangle < \epsilon.$$

Hence, it suffices to consider operators defined by $\langle \nu, Tf \rangle = \int \Psi(y,\bar{y}) \langle d\bar{e}\nu, Sfde \rangle$ where Ψ is continuous, taking values between 0 and 1.

Because Ψ is continuous, there are, for given $\epsilon > 0$, finite collections of components $\{e_i\}$ and $\{\bar{e}_j\}$ and real numbers $r_{i,j}$ that satisfy

$$\int | \Psi - \sum_{i,j} r_{i,j} e_i \otimes \bar{e}_j | \langle d\bar{e}\nu, S\mathbf{1}de \rangle < \epsilon.$$

We conclude that we may assume $\Psi = e \otimes \bar{e}$ for $e, \bar{e} \in \mathcal{E}$. We will need the following lemma.

2.2 lemma. Given $e \in \mathcal{E}$, $\mu \in C"_+$, and $\epsilon > 0$, there are elements $e_1 \in usc \cap \mathcal{E}$ and $e_2 \in lsc \cap \mathcal{E}$ that satisfy $(e_1)_\mu \leqq e \leqq (e_2)_\mu$, $e_1 \leqq e_2$, and $\langle \mu, e_2 - e_1 \rangle < \epsilon$.

Proof. Since $C''_\mu = U_\mu$, we may choose $\dot{e} \in U_+$ with $\dot{e}_\mu = e_\mu$. We may also assume that $\dot{e}$ is a component, replacing $\dot{e}$ with $\vee_n(n\dot{e}\wedge\mathbf{1})$ if necessary (recall that U is σ-closed). Because $\dot{e}$ is the supremum of a subset of usc, we find $f \in$ usc with $0 \leqq f \leqq \dot{e}$ and $\langle \mu, \dot{e} - f\rangle < \epsilon/(4\|\mu\|)$. Consider $A = \{x \in X;\ f(x) \geqq \epsilon/(4\|\mu\|)\}$. Since $f \in$ usc, this set is closed in X. A determines element $e_1 \in \mathrm{usc} \cap \mathcal{E}$ with $\mu(A) = \langle \mu, e_1\rangle$. We have $e_1 \leqq \dot{e}$ and

$$\langle \mu, \dot{e} - e_1\rangle \leqq \langle \mu, \dot{e} - f\rangle + \langle \mu, (\epsilon/(4\,\|\,\mu\,\|))\mathbf{1}\rangle \ < \ \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \frac{1}{2}\epsilon.$$

If we apply the above to $\mathbf{1}$-$\dot{e}$, we find characteristic $(\mathbf{1}-e_2) \in$ usc with

$$\langle \mu, e_2 - \dot{e}\rangle \ = \ \langle \mu, \mathbf{1} - \dot{e} - (\mathbf{1} - e_2)\rangle \ < \ \frac{1}{2}\epsilon.$$

and $(\mathbf{1}-\dot{e}) \geqq (\mathbf{1}-e_2)$. We have then $e_2 \in$ lsc, $e_1 \leqq \dot{e} \leqq e_2$, and

$$\langle \mu, e_2 - e_1\rangle \ = \ \langle \mu, e_2 - \dot{e}\rangle \ + \ \langle \mu, \dot{e} - e_1\rangle \ < \ \frac{1}{2}\epsilon \ + \ \frac{1}{2}\epsilon \ = \ \epsilon.$$

We return to the operator defined by $\int e \otimes \bar{e}\langle d\bar{e}\nu, Sf de\rangle$. Given $\epsilon > 0$, choose $e_1, e_2, \bar{e}_1$, and $\bar{e}_2$ according to the lemma such that

$$\langle S^t\mu, e_2 - e_1\rangle < \frac{1}{2}\epsilon$$

$$\langle \mu, \bar{e}_2 - \bar{e}_1\rangle < \epsilon/(2\,\|\,S\,\|).$$

We next find $f, \bar{f} \in C$ such that $e_1 \leqq f \leqq e_2$ and $\bar{e}_1 \leqq \bar{f} \leqq \bar{e}_2$, since e_1 and $\bar{e}_1$ are from usc and e_2 and $\bar{e}_2$ are from lsc. It follows that

$$\int |f \otimes \overline{f} - e \otimes \bar{e}| \langle d\bar{e}\mu, S1de\rangle \leqq \int (e_2 \otimes \bar{e}_2 - e_1 \otimes \bar{e}_1)\langle d\bar{e}\mu, S1de\rangle$$
$$= \int e_2 \otimes \bar{e}_2 \langle d\bar{e}\mu, S1de\rangle - \int e_1 \otimes \bar{e}_1 \langle d\bar{e}\mu, S1de\rangle$$
$$= \langle \bar{e}_2\mu, Se_2\rangle - \langle \bar{e}_1\mu, Se_1\rangle$$
$$= \langle \bar{e}_2\mu, Se_2\rangle - \langle \bar{e}_1\mu, Se_2\rangle + \langle \bar{e}_1\mu, Se_2\rangle - \langle \bar{e}_1\mu, Se_1\rangle$$
$$\leqq \| S \| \langle \mu, \bar{e}_2 - \bar{e}_1\rangle + \langle S^t\mu, e_2 - e_1\rangle$$
$$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

If the operator R is defined by $\langle \nu, Rg\rangle = \int f \otimes \overline{f}\langle d\bar{e}\nu, Sgde\rangle$, the above shows that $\| R - T \|_\mu < \epsilon$. We now demonstrate that $R \in L^r(C,E)$.

<u>2.3 lemma.</u> If $f \in C_+$, there is a net $f_\alpha \uparrow f$ in order and norm, where each f_α may be written

$$f_\alpha = \sum_{i=1}^{m_\alpha} a_{\alpha,i} e_{\alpha,i}$$

with $e_{\alpha,i} \in lsc \cap \mathcal{E}$.

<u>Proof.</u> Assume, without loss of generality, that $0 \leqq f \leqq 1$. For each $g \in C$ and real number $0 \leqq a \leqq 1$, let $e(a,g) = \vee_n(\mathbf{1} \wedge n(g - a\mathbf{1})^+)$.

By definition, $e(a,g) \in lsc \cap \mathcal{E}$, and $ae(a,g) \leqq g$. In general if

$$0 = a_0 \leqq a_1 \ldots \leqq a_n = 1.$$

let

$$e_i + e(a_i - a_{i-1}, (f - a_{i-1}\mathbf{1})^+) = \vee_n(\mathbf{1} \wedge n((f - a_{i-1}\mathbf{1})^+ - (a_i - a_{i-1})\mathbf{1})^+)$$

we have

$$\sum_{l}^{n}(a_i - a_{i-1})e_i \leqq f$$

and

$$0 \leqq f - \sum_{1}^{n} (a_i - a_{i-1}) e_i \leqq \left(\bigvee_{1}^{n} \mid a_i - a_{i-1} \mid\right) 1.$$

Hence, f is the supremum of all such sums. To find an increasing net simply consider all finite suprema of the sums above. This completes the proof.

We apply the lemma to find $f_\alpha \uparrow f$ and $\bar{f}_\alpha \uparrow \bar{f}$, and consequently $f_\alpha \otimes \bar{f}_\alpha \uparrow f \otimes \bar{f}$. Let one of these be written

$$f_\alpha \otimes \overline{f}_\alpha = \sum_{i,j} \eta_{i,j} e_i \otimes \bar{e}_j.$$

Suppose first that E = C. Let $\{\nu_\beta\}$ be a net in C' which σ(C',C) converges to ν. For $h \in C_+$, $he_i \in$lsc and we may find a net $\{h_\gamma\}$ in C_+ with $h_\gamma \uparrow he_i$. Note that

$$\int e_i \otimes \bar{e}_j \langle d\bar{e}\nu_\beta, Shde \rangle = \langle \bar{e}_j \nu_\beta, She_i \rangle.$$

and we have

$$\langle \bar{e}_j \nu, Sh_\gamma \rangle \leqq liminf_\beta \langle \bar{e}_j \nu_\beta, Sh_\gamma \rangle \leqq liminf_\beta \langle \bar{e}_j \nu_\beta, She_i \rangle.$$

Thus $\langle \bar{e}_j \nu, She_i \rangle \leqq liminf_\beta \langle \bar{e}_j \nu_\beta, Shei \rangle$, and we conclude that the operator defined by $\langle \nu, R'h \rangle = \int e_i \otimes \bar{e}_j \langle d\bar{e}\nu, Shde \rangle$ maps $h \in C_+$ to an element of lsc, and hence the same holds for R_α defined by $\langle \nu, R_\alpha h \rangle = \int f_\alpha \otimes \overline{f}_\alpha \langle d\bar{e}\nu, Shde \rangle$. Since $f_\alpha \otimes \overline{f}_\alpha \uparrow f \otimes \overline{f}$ in order and norm, we have that $\langle \nu, Rh \rangle = \int f \otimes \overline{f} \langle d\bar{e}\nu, Shde \rangle$ maps $h \in E_+$ to an element of lsc.

In an analogous manner, we can show that $Rh \in$usc for $h \in C_+$, and thus $R \in L^b$(C,C), because C is Dedekind closed (i.e. usc$\cap$lsc=C).

Next assume E = U. Fix $h \in C_+$ and consider

$$\begin{aligned} \int f_\alpha \otimes \overline{f}_\alpha \langle d\bar{e}\nu, Shde \rangle &= \sum_{i,j} \eta_{i,j} \int e_i \otimes \bar{e}_j \langle d\bar{e}\nu, Shde \rangle \\ &= \sum_{i,j} \eta_{i,j} \langle \bar{e}_j \nu, She_i \rangle \end{aligned}$$

as a function of ν. For fixed i and j, there is a net $h_\gamma \uparrow he_i$ with $h_\gamma \subset C$. $\langle \bar{e}_j \nu, Sh_\gamma \rangle$ is then, as a function of ν, an element of U. Thus $\int f \otimes \overline{f} \langle d\bar{e}\nu, Shde \rangle$ is a supremum of elements of U (as a function of ν). In a similar manner we can show that $\int f \otimes \overline{f} \langle d\bar{e}\nu, Shde \rangle$ is an infimum of elements of U. Since U is Dedekind closed, the operator above maps C to U.

Every positive operator in the ideal generated by $L^r(C,E)$ is dominated by an operator in $L^r(C,E)$, for if S, T $\geqq 0$, we have $S \vee T \leqq S+T$. As a consequence, the preceding argument implies that $L^r(C,E)$ is dense under $\| \bullet \|_\mu$ in the ideal which it generates in $L^c(C'',C'')$. If $T \geqq 0$ is in the band generated by $L^r(C,C)$, there is a net T_α in the ideal generated by $L^r(C,C)$ with $T_\alpha \uparrow T$. Since $\langle \mu, T_\alpha \mathbf{1} \rangle \uparrow \langle \mu, T\mathbf{1} \rangle$, we conclude $L^r(C,C)$ is dense in the band it generates under $\| \bullet \|_\mu$, and the proof of 2.1 is complete.

We note without proof that if X is a compact metric space, $L^r(C,C)$ may be replaced by $L^b(C,C)$.

2.4 Proposition. Given $T \in L^c(C'',C'')$ and $S \in L^r(C,E)$, E=C or U, with $0 \leqq T \leqq S$, and $\mu \in C'_+$, $T_\mu = \lim_n T^n_\mu$ for a sequence $\{T^n\} \subset L^r(C,E)$ satisfying $0 \leqq T^n \leqq S$.

Proof. By 2.1, there is a sequence $\{T^n\} \subset L^r(C,E)$ satisfying $0 \leqq T^n \leqq S$ and

$$\| T^n - T \| \mu = \langle \mu, | T^n - T | \mathbf{1} \rangle \leqq (\frac{1}{2})^n.$$

Let $S^n = \vee_{m \geq n} T^m$. It follows that

$$\begin{aligned} \langle \mu, | S^n - T | \mathbf{1} \rangle &\leqq \langle \mu, \vee_{m \geq n} | T^m - T | \mathbf{1} \rangle \\ &= \langle \mu, \bigvee_k \bigvee_{m=n}^{k} | T^m - T | \mathbf{1} \rangle \\ &= \bigvee_k \langle \mu, \bigvee_{m=n}^{k} | T^m - T | \mathbf{1} \rangle \end{aligned}$$

The last step follows from the fact that the finite suprema are increasing. Thus

$$\begin{aligned}\langle\mu,\mid S^n - T \mid \mathbf{1}\rangle &\leqq \bigvee_k \sum_{m=n}^{k} \langle\mu,\mid T^m - T \mid \mathbf{1}\rangle \\ &\leqq \bigvee_k \sum_{m=n}^{k} \left(\frac{1}{2}\right)^m \\ &= \left(\frac{1}{2}\right)^{n-1}.\end{aligned}$$

Thus $\lim_n \langle\mu,\mid S^n - T \mid \mathbf{1}\rangle = 0$.

Since $\mathrm{limsup}_m \mathrm{T}^m = \wedge_m \mathrm{S}^m$, we have

$$\lim_{\mathrm{n}} \langle\mu,\mid \mathrm{limsup}_{\mathrm{m}} \mathrm{T}^{\mathrm{m}} - \mathrm{S}^{\mathrm{n}} \mid \mathbf{1}\rangle = 0.$$

From

$$\langle\mu,\mid \mathrm{limsup}_{\mathrm{m}} \mathrm{T}^{\mathrm{m}} - \mathrm{T} \mid \mathbf{1}\rangle \leqq \langle\mu,\mid \mathrm{limsup}_{\mathrm{m}} \mathrm{T}^{\mathrm{m}} - \mathrm{S}^{\mathrm{n}} \mid \mathbf{1}\rangle + \langle\mu,\mid \mathrm{S}^{\mathrm{n}} - \mathrm{T} \mid \mathbf{1}\rangle$$

we obtain

$$\langle\mu,\mid \mathrm{limsup}_{\mathrm{m}} \mathrm{T}^{\mathrm{m}} - \mathrm{T} \mid \mathbf{1}\rangle = 0.$$

Finally, since $(\mathrm{limsup}_m T^m)_\mu = \mathrm{limsup}_m T^m_\mu$, we conclude that $\mathrm{limsup}_m T^m_\mu = \mathrm{T}_\mu$.

In an analogous manner, we can demonstrate that $\mathrm{liminf}_m T^m_\mu = \mathrm{T}_\mu$, and the proposition follows.

Note that $\mathrm{T}_\mu \uparrow \mathrm{T}$ as μ increases in C'_+, with $\{\mathrm{T}_\mu\}$ a net indexed by the directed set C'_+.

<u>2.5 Proposition.</u> If $\mathrm{T} \in \mathrm{L}^c(\mathrm{C}'',\mathrm{C}'')$ with $0 \leqq \mathrm{T} \leqq \mathrm{S} \in \mathrm{L}^r(\mathrm{C},\mathrm{E})$, E=C or U, then T is in the order closure of $\mathrm{L}^r(\mathrm{C},\mathrm{E})$. In particular, if $(\mathrm{L}^r(\mathrm{C},\mathrm{E}))^{\ell u}$ represents the set of all infima of operators which are suprema of subsets of $\mathrm{L}^r(\mathrm{C},\mathrm{E})$, then T is the order limit of a net in $(\mathrm{L}^r(\mathrm{C},\mathrm{E}))^{\ell u}$.

Proof. By 2.4 there is, for each $\mu \in C'_+$, a sequence $\{T^n\} \subset L^r(C,E)$ with $0 \leqq T^n \leqq S$ and $\lim_n T^n_\mu = T_\mu$. In addition, $\limsup_n T^n \leqq S$ and $(\limsup_n T^n)_\mu = T_\mu$. Let $R(\mu) = \limsup_n T^n$. Then $T_\mu \leqq R(\mu) \leqq T + S^d_\mu$, where S^d_μ is the projection of S on $(C"_\mu)^d$. As μ increases in C'_+, $T_\mu \uparrow T$ and $S^d_\mu \downarrow 0$, so that $\lim_\mu R(\mu) = T$.

2.6 Theorem. Let E = C or U. $L^r(C,E)$ is order dense in the band of $L^c(C",C")$ which it generates. If X is a compact metric space, $L^r(C,U)$ is order dense in $L^c(C",C")$.

Proof. The first statement follows immediately from the preceeding arguments. In order to prove the second, we need the following.

2.7 Proposition. Let $T \in L^c(C",C")$ with range in $C"_\mu$ for some $\mu \in C'_+$. If X is a metric space, there is an operator $S \in L^r(C,U)$ such that $S_\mu = T$. If $T \geq 0$, we may choose $S \geq 0$.

Proof. By the lifting theorem (1.1), there is an isometry $I: C"_\mu \to U + N_\mu$ satisfying $(If)_\mu = f$ for $f \in C"_\mu$ and $I\mathbf{1}_\mu = \mathbf{1}$. If X is a metric space, C(X) is separable. Let $\{f_n\}$ be a countable norm dense subset of C(X). $\{Tf_n\}$ is norm dense in $T(C(X)) \subset C"$. Let $g_n = I(Tf_n)$. We may write $g_n = h_n + k_n$ with $h_n \in U$ and $k_n \in N$. In addition, we may choose k_n which satisfy $|k_n| \leqq j_n$ for some $j_n \in C"^d_\mu \cap U$. Let $\mathbf{1}_n = \vee_m (mj_n) \wedge \mathbf{1}$ and $\mathbf{1}_A = \vee_n \mathbf{1}_n$. It follows that $\mathbf{1}_n \in C"^d_\mu \cap U$ and thus $\mathbf{1}_A \in C"^d_\mu \cap U$, since U is σ-closed.

Define I' by $I' = (\mathbf{1}-\mathbf{1}_A)I$. I' maps Tf_n to U for each n, since

$$(\mathbf{1} - \mathbf{1}_A)g_n = (\mathbf{1} - \mathbf{1}_A)h_n + (\mathbf{1} - \mathbf{1}_A)k_n = (\mathbf{1} - \mathbf{1}_A)h_n.$$

Because $\mathbf{1}-\mathbf{1}_A \in C"^d_\mu \cap U$, we have $(I'(Tf_n))_\mu = Tf_n$. From the fact that $\{Tf_n\}$ is norm dense in $T(C(X))$ and I' is norm continuous, we conclude that $I'(T(C)) \subset U$, since U is norm closed. $\mathbf{1}-\mathbf{1}_A \in C"^d_\mu \cap U$ implies that $(I'(Tf))_\mu = Tf$ for all $f \in C$.

I' composed with T restricted to C determines a map $\overline{S} \in L^b(C,C")$ with image in U, which defines $S \in L^r(C,U)$. Note that $(Sf)_\mu = Tf$ for $f \in C$. If $f \in C"$ is arbitrary, we have $(Sf)_\mu = Tf$, because S and T are order continuous and C is dense in C".

REFERENCES

1. A. Ionescu Tulcea and C. Ionescu Tulcea, Topics in the Theory of Lifting, Berlin-Heidelberg-New York:Springer 1969.

2. D. Carothers, "Order and Closure Properties of Order Continuous Operators", Ph.D. dissertation, Purdue University (1981).

3. S. Kaplan, The Bidual of C(X)I, Amsterdam:Elsevier Science Publishers 1985.

4. S. Kaplan, "Closure Properties of C(X) in its Second Dual", Proc. Amer. Math. Soc. 17 (1966) 401-406.

5. H. Nakano, "Product Spaces of Semi-Ordered Linear Spaces", J. Fac. Sci. Hokkaido Univ. Ser. I 12 (1953) 163-210.

6. H.H. Schaefer, Banach Lattices and Positive Operators, New York-Heidelberg-Berlin: Springer-Verlag 1974.

7. B.C. Vulikh, Introduction to the Theory of Partially Ordered Vector Spaces, Groningen: Wolters-Noordhoff 1967.

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