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BILINEAR INTEGRATION OF AN EXTREME POINT MULTIFUNCTION

By using a generalized theorem of Lebesgue-Nikodym, we show that under suitable restrictions the bilinear integrals of a multifunction and the corresponding extreme point multifunction are equal. Examples are given showing that the hypotheses of the main theorems cannot be weakened.

1. INTRODUCTION

Various developments in mathematical economics, control theory and statistics have led to the study of measurable and integrable multifunctions. A beginning of what might be called a calculus of multifunctions can be found in R.J. Aumann's influential paper [2]. The integration of multifunctions has been studied extensively in recent years by numerous authors. The foundations were laid by R.J. Aumann [2], C. Castaing [7], G. Debreu [10], K. Kuratowski and C. Ryll-Nardzewski [19], C. Olech [20], A. Pliś [22], and others. C. Castaing [8] and C.J. Himmelberg and F.S. van Vleck [16] showed that under suitable restrictions the measurability of a multifunction F implies the measurability of the multifunction $\text{ext}\ F$, where $\text{ext}\ F(t)$ is the set of extreme points of $F(t)$. The main purpose of this paper is to show, by using a generalized theorem of Lebesgue-Nikodym, that the bilinear integrals (in the sense of N. Dinculeanu [11]) of these two multifunctions are equal. This extends corresponding results on the same topic, see remarks 4.8.

Section 2 of this paper is reserved for some preliminary notes concerning notation, definition and remarks. The general reference for measure theoretic properties and the bilinear integral is [11].

In Section 3 we present some results, most of which are known and stated therefore without proof. All of these will be used later.

Section 4 is concerned with the integration of multifunctions. The main results are theorems 4.3 and 4.7.

Section 5 is devoted to some examples showing that the hypotheses of the main theorems cannot be weakened.

2. PRELIMINARIES

Throughout this paper T will denote a non-empty point set on which no topological structure is required.

2.1 MEASURES

Let V be a fixed Banach space and $\mathcal{C}$ a ring of subsets of T. Recall that a δ-ring is a ring which is closed under countable intersections. Let $m: \mathcal{C} \to V$ be a measure. For every set $A \in \mathcal{C}$, let

$$|m|(A) = \sup_J \sum_{j \in J} \|m(A_j)\|,$$

where the supremum is extended over all finite classes $\{A_j | j \in J\}$ of disjoint sets of $\mathcal{C}$ such that $A = \bigcup_{j \in J} A_j$. The number $|m|(A)$ is called the *variation* of m on the set A. The function $|m|$ is called the *variation* of m on $\mathcal{C}$. If $|m|(A) < \infty$ for every $A \in \mathcal{C}$, then $|m|$ is of finite variation with respect to $\mathcal{C}$. It is known that $|m|$ is a finite measure on $\mathcal{C}$. Extend the finite measure $|m|$ on $\mathcal{C}$ by means of the Carathéodory method to a measure $|m|^*$ on the σ-algebra $\mathcal{P}(|m|)$ of all $|m|$*-measurable* sets. The class $\Sigma(|m|) = \{E \in \mathcal{P}(|m|) \,|\, |m|^*(E) < \infty\}$ is the δ-ring of all $|m|$*-integrable* sets. The restriction of $|m|^*$ to $\Sigma(|m|)$ is denoted by $|m|$. If $E \in \Sigma(|m|)$ and $|m|(E) = 0$, then E is called $|m|$*-negligible*.

2.1.1 DEFINITION ([11], p. 179). Denote by $C(|m|)$ the collection of all classes $\mathcal{A} = \{A_i | i \in I\}$ of disjoint $|m|$-integrable sets such that $T - \bigcup_{i \in I} A_i$ is $|m|$-negligible and such that for every set $A \in \mathcal{C}$ there exists an $|m|$-negligible set $N \subset A$ and an at most countable set $J \subset I$ with $A - N = \bigcup_{i \in J} (A \cap A_i)$. We say that the measure $|m|$ has the *direct* *sum property* if $C(|m|) \neq \emptyset$. A measure of finite variation is said to have the direct sum property if its variation has this property.

2.1.2 SOME PROPERTIES. (a) The measure m on $\mathcal{C}$ can be extended to a measure, again denoted by m, on $\Sigma(|m|)$ (see [11], p. 76).

(b) If $\mathcal{C}$ is a σ-algebra and if $|m|$ is complete on $\mathcal{C}$, then $\mathcal{C} = \Sigma(|m|) = \mathcal{P}(|m|)$. (See Section 5).

(c) If T is a countable union of sets of $\mathcal{C}$, then $|m|^*$ is a σ-finite and complete measure on $\mathcal{P}(|m|)$. Thus $|m|$ on $\Sigma(|m|)$ is also complete.

(d) Whenever m is supposed to be non-atomic, it must be understood that m is non-atomic on $\Sigma(|m|)$, that is, the extended measure m is non-atomic. This convention is necessary, because the extension of a non-atomic measure need not be non-atomic, see [5], p. 2 or [30], p. 67 for examples.

(e) If m is non-atomic on $\Sigma(|m|)$, so is $|m|$.

(f) If T is a countable union of sets of the δ-ring $\Sigma(|m|)$, then m has the direct sum property. This follows from the fact that $\mathcal{C} \subset \Sigma(|m|)$.

2.2 MEASURABILITY

Throughout this paper U will denote a Banach space.

A function $f\colon T \to U$ is *$|m|$-measurable* if $f^{-1}(C) \in \mathcal{P}(|m|)$ for every closed set C in U.

A multifunction $F\colon T \to U$ is a function whose domain is T and whose values are non-empty subsets of U. If $A \subset U$, then $F^-(A) = \{t \in T \mid F(t) \cap A \neq \emptyset\}$.

A multifunction $F\colon T \to U$ is *$|m|$-measurable* (*weakly $|m|$-measurable*) if $F^-(A) \in \mathcal{P}(|m|)$ for every closed (open) subset A of U. This definition of $|m|$-measurability was formalized by C. Castaing [7], while the term "weak measurability" was introduced by C.J. Himmelberg, M.Q. Jacobs and F.S. van Vleck [15].

A function $f\colon T \to U$ is called a *selector* for F if $f(t) \in F(t)$ $|m|$-a.e. on T. The set of all $|m|$-measurable selectors of F will be denoted by $\mathcal{S}_F$. If $f \in \mathcal{S}_F$, consider the equivalence class $\tilde{f} = \{g\colon T \to U \mid g(t) = f(t)\ |m|\text{-a.e. on } T\}$. Write $S_F = \{\tilde{f} \mid f \in \mathcal{S}_F\}$.

Following R.T. Rockafellar [25], we way that a multifunction $F: T \to U$ admits a *Castaing representation* if there exists a countable set $M = \{f_i | i \in I\} \subset S_F$ such that $M(t) = \{f_i(t) | i \in I\}$ is dense in $F(t)$ $|m|$-a.e. on T. (See [7], p. 116).

2.3 INTEGRABILITY

Let W be a third fixed Banach space and consider a bilinear transformation $(u,v) \to uv$, defined on $U \times V$ into W such that $\|(u,v)\| \leq \|u\| . \|v\|$.

The vector integral being employed is the "bilinear" or "m-integral" of Dinculeanu. Let

$$E_U(\Sigma(|m|)) = \{f: T \to U | f = \sum_{i \in I} x_i \chi_{A_i},\ x_i \in U,\ A_i \in \Sigma(|m|) \text{ and } I \text{ is a finite index set}\}.$$

If $f = \sum_{i \in I} x_i \chi_{A_i} \in E_U(\Sigma(|m|))$, then $\int f(t)\, dm = \sum_{i \in I} x_i m(A_i) \in W$, $|f| = \sum_{i \in I} \|x_i\| \chi_{A_i}$ and $\|f\|_E = \int |f|(t)\, d|m|$. The semi-norm $\|\cdot\|_E$ defines on the vector space $E_U(\Sigma(|m|))$ the *topology of the convergence in mean.*

A function $f: T \to U$ is m-*integrable* if there exists a Cauchy sequence (f_n) in $E_U(\Sigma(|m|))$ such that $f_n \to f$ $|m|$-a.e. on T. Then $\int f(t)dm \in W$.

The space of all m-integrable functions $f: T \to U$ will be denoted by $\mathcal{L}^1_U(m)$. The set of all $|m|$-integrable selectors of $F: T \to U$ will be denoted by I_F. Then $I_F \subset S_F$. Write $I_F = \{\tilde{f} | f \in I_F\}$. If $f \in \mathcal{L}^1_U(m)$ and $A \in P(|m|)$, then $f\chi_A \in \mathcal{L}^1_U(m)$ and $\int_A f(t)dm = \int f(t)\chi_A(t)dm$. If $A \in P(|m|)$, then the integral of a multifunction $F: T \to U$ over A is defined by

$$\int_A F(t)dm = \{\int_A f(t)dm | f \in I_F\}.$$

We observe that $\int_A F(t)dm$ exists, even if F is not $|m|$-measurable. Furthermore, $\int_A F(t)dm$ may be empty, even if $U = \mathbb{R}$.

2.3.1 SOME PROPERTIES. (a) If f is m-integrable, then f is $|m|$-measurable. This follows from the definition of m-integrability and the theorem of Egorov [11], p. 94.

(b) [11], p. 122: If $f \in \mathcal{L}^1_U(m)$ and $A \in P(|m|)$, then $f\chi_A \in \mathcal{L}^1_U(m)$.

(c) [11], p. 125: Let $f,g \in \mathcal{L}^1_U(m)$. Then $\int \|f(t) - g(t)\| d|m| = 0$ if and only if $f(t) = g(t)$ $|m|$-a.e. In this case $\int f(t)dm = \int g(t)dm$.

(d) $\mathcal{L}^p_U = \mathcal{L}^p_U(|m|) = \{f\colon T \to U \,|\, f$ is $|m|$-measurable and $\|f(\cdot)\|^p \in \mathcal{L}^1_{\mathbb{R}}(m)\}$.

Also, [11], p. 218: $f \in \mathcal{L}^p_U$ if and only if f is $|m|$-measurable and $\|f(\cdot)\| \in \mathcal{L}^p_{\mathbb{R}}$.

(e) [11], p. 218: If $f\colon T \to U$ is $|m|$-measurable and if there exists a positive function $g \in \mathcal{L}^p_{\mathbb{R}}$ such that $\|f(t)\| \leqslant g(t)$ $|m|$-a.e. on T, then $f \in \mathcal{L}^p_U$.

A multifunction $F\colon T \to U$ is said to be *p-integrably bounded*, $1 \leqslant p < \infty$, if there exists a $k \in \mathcal{L}^p_{\mathbb{R}}(|m|)$ such that

$$\sup\{\|u\| \,|\, u \in F(t)\} \leqslant k(t) \quad |m|\text{-a.e. on } T.$$

If $F\colon T \to U$ is 1-integrably bounded by $k \in \mathcal{L}^1_{\mathbb{R}}(|m|)$, we say that F is integrably bounded by k.

2.4 SCALARWISE MEASURABILITY

Let P be a property possessed by some subsets of the Banach space U. A multifunction $F\colon T \to U$ is said to be *point*-P if for every $t \in T$, $F(t)$ has property P. Denote the topological dual of U by U'.

Following M. Valadier [28], we say that the point-compact convex multifunction $F\colon T \to U$ is *scalarwise* $|m|$*-measurable* (*-integrable*) if for every $x' \in U'$, the function $h_{x'}\colon T \to \mathbb{R}$, defined by

$$h_{x'}(t) = \sup\{\langle x,x'\rangle \,|\, x \in F(t)\}$$

is $|m|$-measurable (-integrable).

2.5 SIMPLE AND WEAK MEASURABILITY

The definitions given here are all from [11] and are stated in general terms.

2.5.1 <u>DEFINITION</u>. If X is a Banach space and Z a subspace of X', then Z is said to be a *norming subspace* of X' if

$$\|x\| = \sup\left\{\frac{|< x,z >|}{\|z\|} \mid z \in Z,\ z \neq 0\right\}, \quad \text{for every } x \in X.$$

Then, X can be imbedded isometrically in Z'.

If X and Y are linear spaces, then the space of all linear transformations from X to Y will be denoted by $L^*(X,Y)$.

2.5.2 <u>DEFINITION</u>. Let X and Y be Banach spaces. We say that a function $U\colon T \to L^*(X,Y)$ is *simply* $|m|$*-measurable*, if for every $x \in X$ the function $\phi_x\colon T \to Y$, defined by $\phi_x(t) = U(t)x$, is $|m|$-measurable.

2.5.3 <u>DEFINITION</u>. Let X and Y be Banach spaces and $Z \subset Y'$ a norming subspace. We say that a function $U\colon T \to L^*(X,Y)$ is Z*-weakly* $|m|$*-measurable*, if for every $x \in X$ and every $z \in Z$, the function $\phi_{x,z}\colon T \to \mathbb{R}$, defined by $\phi_{x,z}(t) = < U(t)x,z >$, is $|m|$-measurable.

For the properties of simply and Z-weakly $|m|$-measurable functions, we refer to [11], pp. 101 - 106.

2.6 <u>OTHER NOTATIONS</u>

Denote by $\mathcal{B}_U$ the Borel σ-algebra of U and by $\mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$ the σ-algebra generated by the class

$$\mathcal{P}(|m|) \times \mathcal{B}_U = \{A \times B \mid A \in \mathcal{P}(|m|),\ B \in \mathcal{B}_U\}.$$

The graph of the multifunction $F\colon T \to U$ is the set

$$\mathcal{G}(F) = \{(t,u) \in T \times U \mid u \in F(t)\}.$$

A topological space is *Polish* if it is separable and metrizable by a complete metric; it is *Suslin* if it is metrizable and the continuous image of a Polish space.

If $F\colon T \to U$ is a multifunction, then the multifunction ext $F\colon T \to U$, defined for every $t \in T$ by

$$(\text{ext } F)(t) = \{u \in F(t) \mid u \text{ is an extreme point of } F(t)\},$$

is called the *extreme point multifunction* determined by F. Multifunctions will be denoted by the capitals F, G and H. If $A \subset U$, then co A denotes the *convex hull* of A.

3. SOME BASIC RESULTS

We state the following propositions in forms which are adequate for the sequel.

3.1 PROPOSITION ([19], p. 398). Let U be separable and $F: T \to U$ point-closed and weakly $|m|$-measurable. Then F has an $|m|$-measurable selector.

3.2 COROLLARY. Let U be separable and $F: T \to U$ point-closed and $|m|$-measurable. Then F has an $|m|$-measurable selector.

PROOF. If O is open in U, then $O = \bigcup_{n=1}^{\infty} C_n$, where the C_n are all closed in U. Then $F^{-}(O) = \bigcup_{n=1}^{\infty} F^{-}(C_n) \in \mathcal{P}(|m|)$. Thus, F is weakly $|m|$-measurable and proposition 3.1 holds. ∇

3.3 PROPOSITION ([29], p. 868). Let T be a countable union of sets of the ring $\mathcal{C}$, U separable and $F: T \to U$ point-closed. Then the following conditions are equivalent:

(1) F is $|m|$-measurable;

(2) F is weakly $|m|$-measurable;

(3) $\mathcal{G}(F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$;

(4) F admits a Castaing representation.

Note that the assumption on T implies completeness of the measure space $(T, \mathcal{P}(|m|), |m|^*)$, see 2.1.2(c). This in turn implies that $\mathcal{P}(|m|)$ is a Suslin family (see [27], p. 50 or [29], p. 864), as is required for proposition 3.3 to hold. It is possible to show by means of a suitable example that the completeness of $(T, \mathcal{P}(|m|), |m|^*)$ is indeed necessary, see for example [1], p. 27. A further requirement in [29], p. 868 is that U be Suslin, which it surely is since it is Polish. These remarks also apply to the proposition below, originally proved for a complete measurable space, a Suslin space U and where F need neither be closed-valued nor $|m|$-measurable. This proposition is a generalization of the so-called Von Neumann-Aumann selection theorem, see [3] or [21], p. 69.

3.4 PROPOSITION ([26], p. 7.11). Let T be a countable union of sets of $\mathcal{C}$, U separable and $F: T \to U$ such that $\mathcal{G}(F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$. Then F has an $|m|$-measurable selector.

3.5 PROPOSITION. If $F: T \to U$ is point-compact convex and $|m|$-measurable, then F is scalarwise $|m|$-measurable.

PROOF. The function $h_{x'}: T \to \mathbb{R}$ defined in 2.4 is $|m|$-measurable, see [9], lemma 5, p. 231. Consequently, F is scalarwise $|m|$-measurable. ∇

3.6 PROPOSITION ([12], p. 439). A non-empty compact subset of a locally convex linear topological Hausdorff space has extreme points.

We now employ a theorem of M. Benamara [4] which deals with

(i) a point-compact convex $F: T \to U'$ which is scalarwise $|m|$-measurable, i.e. if for every $x \in U$, the function $h_x: T \to \mathbb{R}$, defined by

$$h_x(t) = \sup\{\langle x', x \rangle \mid x' \in F(t)\}$$

is $|m|$-measurable;

(ii) a complete measure space.

With remark 2.1.2(c) in mind, we now have:

3.7 PROPOSITION ([4], p. 1249). Let T be a countable union of sets of the ring $\mathcal{C}$, U separable and $F: T \to U'$ point-$\sigma(U',U)$-compact convex and scalarwise $|m|$-measurable. Then the set $\text{ext}\ \mathcal{S}_F$ of all extreme points of $\mathcal{S}_F$ is non-empty and equal to the set $\mathcal{S}_{\text{ext}\ F}$.

3.8 PROPOSITION ([16], p. 725). If $F: T \to \mathbb{R}^n$ is point-compact convex and $|m|$-measurable, then $\mathcal{G}(\text{ext}\ F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_{\mathbb{R}^n})$. Furthermore, if T is a countable union of sets of the ring $\mathcal{C}$, then $\text{ext}\ F$ is $|m|$-measurable.

3.9 PROPOSITION. Let (F_n) be a sequence of multifunctions, $F_n\colon T \to U$, with $\mathcal{G}(F_n) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$ for all n. Define the multifunctions $G_i\colon T \to U$, $i = 1,2,3,4$ by the respective equalities $G_1(t) = \bigcup_{n=0}^{\infty} F_n(t)$; $G_2(t) = \bigcap_{n=0}^{\infty} F_n(t)$; $G_3(t) = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} F_k(t)$ and $G_4(t) = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} F_k(t)$. Then we have that

$$\mathcal{G}(G_i) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U), \quad i = 1,2,3,4.$$

PROOF. It is a matter of routine to show that

$$\mathcal{G}(G_1) = \bigcup_{n=0}^{\infty} \mathcal{G}(F_n), \quad \mathcal{G}(G_2) = \bigcap_{n=0}^{\infty} \mathcal{G}(F_n).$$

Consequently, $\mathcal{G}(G_i) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$, $i = 1,2$. The remaining two equalities follow immediately. ∇

3.10 PROPOSITION ([24], pp. 166, 167). If $\dim U < \infty$, then a compact convex subset A of U equals the convex hull of the set of its extreme points, in symbols $A = \operatorname{co} \operatorname{ext} A$.

The two propositions below are stated in general terms.

3.11 PROPOSITION. Let X and Y be linear spaces. If $f \in \mathcal{L}^*(X,Y)$ and if C is a non-empty convex subset of X and B an extreme subset of $f(C)$, then $f^{-1}(B) \cap C$ is an extreme subset of C.

3.12 PROPOSITION (M. Krein and D. Milman [18]). If A is a compact subset of a locally convex linear topological Hausdorff space and E is the set of extreme points of A, then $A \subset \overline{\operatorname{co}}\, E$, where $\overline{\operatorname{co}}\, E$ denotes the closure of the convex hull of E. Consequently, $\overline{\operatorname{co}}\, A = \overline{\operatorname{co}}\, E$. If, in addition, A is convex, then each closed extreme subset of A contains an extreme point of A and $A = \overline{\operatorname{co}}\, E$.

4. MAIN RESULTS

4.1 THEOREM. If U is separable and $F\colon T \to U$ is integrably bounded, point-closed and $|m|$-measurable, then $\int_A F(t)dm \neq \emptyset$ for every $A \in \mathcal{P}(|m|)$.

PROOF. Corollary 3.2 asserts that F has an $|m|$-measurable selector f. If $k \in \mathcal{L}^1_{\mathbb{R}}(|m|)$ is the bounding function, then $\|f(t)\| \leq k(t)$ $|m|$-a.e. By 2.3.1(e), $f \in \mathcal{L}^1_U(m)$. Consequently, $f \in I_F$, and so $\int_A F(t)dm \neq \emptyset$ for every $A \in \mathcal{P}(|m|)$ by 2.3.1(b). ∇

D. Blackwell [6] extended Lyapunov's convexity theorem by proving that the ranges of certain vector integrals with values in $\mathbb{R}^n$ are compact and, in the non-atomic case, convex. The convexity part of Blackwell's theorem was generalized by H. Richter [23]. By keeping proposition 3.12 in mind, we state Richter's theorem in the following form:

4.2 THEOREM ([23], p. 86). (1) If $F: T \to \mathbb{R}^n$ and m is non-atomic, then $\int_A F(t)d|m|$ is convex for every $A \in \Sigma(|m|)$.
(2) Let T be a countable union of sets of $\mathcal{C}$, m non-atomic and $F: T \to \mathbb{R}^n$ integrably bounded, point-compact convex and $|m|$-measurable. Then $\int_A F(t)d|m|$ is compact and convex for every $A \in \Sigma(|m|)$.

4.3 THEOREM. Let T be a countable union of sets of $\mathcal{C}$, m non-atomic and $F: T \to \mathbb{R}^n$ integrably bounded, point-compact convex and $|m|$-measurable. Then

$$\int_A F(t)d|m| = \int_A (\text{ext } F)(t)d|m|$$

for every $A \in \Sigma(|m|)$.

PROOF. (1) Let $A \in \Sigma(|m|)$ be arbitrary. Theorem 4.1 asserts that $\int_A F(t)d|m| \neq \emptyset$ since $S_F \neq \emptyset$. The integrably boundedness of F implies that $S_F = I_F$. By proposition 3.5, F is scalarwise $|m|$-measurable. Since $\dim(\mathbb{R}^n) < \infty$, we may conclude from proposition 3.7 that $S_{\text{ext } F} \neq \emptyset$. Since ext F is also integrably bounded, it follows that $S_{\text{ext } F} = I_{\text{ext } F}$. Consequently, $\int_A (\text{ext } F)(t)d|m| \neq \emptyset$. If $|m|(A) = 0$, we obviously have that

$$\int_A F(t)d|m| = \{0\} = \int_A (\text{ext } F)(t)d|m|.$$

Henceforth, we suppose that $|m|(A) > 0$. Since $\mathcal{C} \subset \Sigma(|m|)$ by our construction in 2.1, it follows from the assumption on T and

2.1.2(f) that $|m|$, and hence m, has the direct sum property. By theorems 4.1 and 4.2, $\int_A F(t)d|m|$ is non-empty, compact and convex. Proposition 3.6 asserts that the set $\text{ext} \int_A F(t)d|m|$ of all extreme points of $\int_A F(t)d|m|$ is non-empty.

(2) Let $x \in \text{ext} \int_A F(t)d|m|$. Then there exists a function $f \in \mathcal{I}_F$ such that $x = \int_A f(t)d|m|$. Furthermore, $\int_A f(t)d|m|$ cannot be written as a proper convex combination of any two distinct members of $\int_A F(t)d|m|$. We show that $\int_E f(t)d|m| \in \text{ext} \int_E F(t)d|m|$ for every set $E \in \Sigma(|m|)$, $E \subset A$. We notice that $\int_E f(t)d|m| \in \int_E F(t)d|m|$ for every set $E \in \Sigma(|m|)$, $E \subset A$. For such E we also have that $\int_E F(t)d|m|$ is compact and convex, consequently $\text{ext} \int_E F(t)d|m| \neq \emptyset$; see part (1) above. Consider an arbitrary subset $E \in \Sigma(|m|)$, $E \subset A$. We suppose that $|m|(E) > 0$. That there are $|m|$-integrable subsets of A with positive measure follows from the non-atomicity of m, hence of $|m|$ (2.1.2(e)). Consider the multifunction $F_E\colon T \to \mathbb{R}^n$ defined by the equality

$$F_E(t) = \begin{cases} F(t) & \text{if} \quad t \in E \\ \{0\} & \text{if} \quad t \in T - E. \end{cases}$$

Let C be an arbitrary closed subset of $\mathbb{R}^n$. If $0 \in C$, then

$$F_E^-(C) = (F^-(C) \cap E) \cup (T - E) \in \mathcal{P}(|m|);$$

if $0 \notin C$, then

$$F_E^-(C) = F^-(C) \cap E \in \mathcal{P}(|m|).$$

Consequently, F_E is $|m|$-measurable. Furthermore, F_E is integrably bounded and point-compact convex. We now have that $I_{F_E} = S_{F_E}$. Corollary 3.2 and theorem 4.1 show that I_{F_E} and $\int F_E(t)d|m|$ are non-empty, respectively. We show that

$$(*) \qquad \int F_E(t)d|m| = \int_E F(t)d|m|.$$

Let $x \in \int F_E(t)d|m|$. Then there exists a function $f_E \in \mathcal{I}_{F_E}$ such that $x = \int f_E(t)d|m|$. Define the multifunction $G\colon T \to \mathbb{R}^n$ by the equality

$$G(t) = \begin{cases} \{f_E(t)\} & \text{if} \quad t \in E \\ F(t) & \text{if} \quad t \in T - E. \end{cases}$$

Then G is clearly $|m|$-measurable and point-compact. Another application of corollary 3.2 shows that G has an $|m|$-measurable selector $g: T \to \mathbb{R}^n$. Then $g \in S_F = I_F$ and by 2.3.1(b), $g\chi_E \in \mathcal{L}^1_{\mathbb{R}^n}$. But, $g(t)\chi_E(t) = f_E(t)$ $|m|$-a.e. on T. By 2.3.1(c) we now have that

$$x = \int f_E(t)d|m| = \int g(t)\chi_E(t)d|m| = \int_E g(t)d|m| \in \int_E F(t)d|m|.$$

Conversely, if $y \in \int_E F(t)d|m|$, then there exists a selector $f \in I_F$ such that $y = \int_E f(t)d|m|$. But $f\chi_E \in I_{F_E}$, and so,

$$y = \int_E f(t)d|m| = \int f(t)\chi_E(t)d|m| \in \int F_E(t)d|m|.$$

We therefore have that

$$\int F_E(t)d|m| = \int_E F(t)d|m|.$$

Suppose now that

$$\int_E f(t)d|m| \notin \text{ext} \int_E F(t)d|m| = \text{ext} \int F_E(t)d|m|.$$

It follows then that there exist two distinct elements $\int g_1(t)d|m|$, $\int g_2(t)d|m| \in \int F_E(t)d|m|$ and an $\alpha \in (0,1)$, where $g_1, g_2 \in I_{F_E}$, such that

$$\int_E f(t)d|m| = \alpha \int g_1(t)d|m| + (1 - \alpha) \int g_2(t)d|m|. \quad \text{(i)}$$

We have that

$$\begin{aligned}\int g_1(t)d|m| &= \int_E g_1(t)d|m| + \int_{T-E} g_1(t)d|m| \\ &= \int_E g_1(t)d|m|. \qquad \text{(ii)}\end{aligned}$$

Similarly,

$$\int g_2(t)d|m| = \int_E g_2(t)d|m|. \quad \text{(iii)}$$

Hence,

$$\int_E g_1(t)d|m| \neq \int_E g_2(t)d|m|.$$

It follows from (i), (ii) and (iii) that

$$\int_E f(t)d|m| = \alpha \int_E g_1(t)d|m| + (1 - \alpha) \int_E g_2(t)d|m|. \quad \text{(iv)}$$

Define the functions $h_i: T \to \mathbb{R}^n$, $i = 1,2$, by

$$h_i(t) = \begin{cases} g_i(t) & \text{if} \quad t \in E \\ f(t) & \text{if} \quad t \in T - E; \quad i = 1,2. \end{cases}$$

The functions h_1 and h_2 are clearly $|m|$-measurable and since F is integrably bounded, we also have that $h_1, h_2 \in \mathcal{L}^1_{\mathbb{R}^n}(|m|)$. Furthermore, $h_1, h_2 \in I_F$. We now have that

$$\int_A h_1(t)d|m| = \int_E h_1(t)d|m| + \int_{A-E} h_1(t)d|m|$$
$$= \int_E g_1(t)d|m| + \int_{A-E} f(t)d|m|,$$

and similarly,

$$\int_A h_2(t)d|m| = \int_E g_2(t)d|m| + \int_{A-E} f(t)d|m|.$$

We deduce that

$$\int_A h_1(t)d|m| \neq \int_A h_2(t)d|m|.$$

Thus, by using (iv), it follows that

$$\int_A f(t)d|m| = \int_E f(t)d|m| + \int_{A-E} f(t)d|m|$$
$$= \alpha \int_E g_1(t)d|m| + \alpha \int_{A-E} f(t)d|m|$$
$$+ (1 - \alpha) \int_E g_2(t)d|m| + (1 - \alpha) \int_{A-E} f(t)d|m|$$
$$= \alpha \int_A h_1(t)d|m| + (1 - \alpha) \int_A h_2(t)d|m|.$$

This shows that

$$\int_A f(t)d|m| \notin \operatorname{ext} \int_A F(t)d|m|,$$

which is an absurdity. Consequently,

$$\int_E f(t)d|m| \in \operatorname{ext} \int_E F(t)d|m|$$

for every $E \in \Sigma(|m|)$, $E \subset A$.

(3) Consider the multifunction $F_A\colon T \to \mathbb{R}^n$ defined by

$$F_A(t) = \begin{cases} F(t) & \text{if} \quad t \in A \\ \{0\} & \text{if} \quad t \in T - A \end{cases}$$

for $A \in \Sigma(|m|)$. We have that

$$\int_A F(t)d|m| = \int F_A(t)d|m|,$$

hence that

$$\operatorname{ext} \int_A F(t)d|m| = \operatorname{ext} \int F_A(t)d|m|.$$

By the same procedure used to establish (*) in part (2), one can show that

$$\int_A (\text{ext } F)(t)d|m| = \int (\text{ext } F_A)(t)d|m| ;$$

the only difference of significance being that we do not apply corollary 3.2 but proposition 3.8 to show that ext F and ext F_A are $|m|$-measurable and proposition 3.7 to show that $S_{\text{ext } F} \neq \emptyset$ and $S_{\text{ext } F_A} \neq \emptyset$. We now show that

$$\text{ext } \int_A F(t)d|m| \subset \int_A (\text{ext } F)(t)d|m|,$$

or equivalently, that

$$\text{ext } \int F_A(t)d|m| \subset \int (\text{ext } F_A)(t)d|m|.$$

Suppose that $x \in \text{ext } \int_A F(t)d|m| - \int_A (\text{ext } F)(t)d|m|$. Then there exists a function $f \in I_F$ such that $x = \int_A f(t)d|m|$. But then

$$x = \int f(t)\chi_A(t)d|m| \in \text{ext } \int F_A(t)d|m|$$

and

$$\int f(t)\chi_A(t)d|m| \notin \int (\text{ext } F_A)(t)d|m|.$$

This means that $f\chi_A \notin I_{\text{ext } F_A}$. Since ext F_A is integrably bounded, we have that $I_{\text{ext } F_A} = S_{\text{ext } F_A}$. For the same reason, $I_{F_A} = S_{F_A}$. By proposition 3.7, we have that

$$I_{\text{ext } F_A} = S_{\text{ext } F_A} = \text{ext } S_{F_A} = \text{ext } I_{F_A}.$$

It now follows that $f\chi_A \notin \text{ext } I_{F_A}$, and hence, $\widetilde{f\chi_A} \notin \text{ext } I_{F_A}$. Since $\widetilde{f\chi_A} \in I_{F_A}$, we deduce that there exist two classes $\tilde{\ell}_1, \tilde{\ell}_2 \in I_{F_A}$, $\tilde{\ell}_1 \neq \tilde{\ell}_2$ and an $\alpha \in (0,1)$ such that

$$\widetilde{f\chi_A} = \alpha\tilde{\ell}_1 + (1 - \alpha)\tilde{\ell}_2.$$

This means that

$$f\chi_A = \alpha\ell_1 + (1 - \alpha)\ell_2$$

where $\ell_1 \in \tilde{\ell}_1$ and $\ell_2 \in \tilde{\ell}_2$. Consequently, ℓ_1 differs from ℓ_2 on a set of positive $|m|$-measure. Since $\ell_1, \ell_2 \in I_{F_A}$, it follows that

$$\int_{T-A} \ell_1(t)d|m| = 0 = \int_{T-A} \ell_2(t)d|m|.$$

We now deduce from [11], p. 188, corollary 2 that there exists a set $E_1 \in \Sigma(|m|)$ where $E_1 \subset A$ and $|m|(E_1) > 0$ such that

$$\int_{E_1} \ell_1(t)d|m| \neq \int_{E_1} \ell_2(t)d|m|.$$

We have that $\ell_1\chi_{E_1}, \ell_2\chi_{E_1} \in I_{F_{E_1}}$, where $F_{E_1} : T \to \mathbb{R}^n$ is the multifunction defined in the same fashion as the multifunction F_E in part (2) of the present proof. It follows that

$$\int f(t)\chi_{E_1}(t)d|m| = \int_{E_1} f(t)d|m| = \alpha \int_{E_1} \ell_1(t)d|m| + (1-\alpha) \int_{E_1} \ell_2(t)d|m|.$$

Consequently,

$$\int_{E_1} f(t)d|m| \notin \text{ ext } \int F_{E_1}(t)d|m| = \text{ ext } \int_{E_1} F(t)d|m|.$$

By what has been achieved in part (2), we deduce that

$$x = \int_A f(t)d|m| \notin \text{ ext } \int_A F(t)d|m|,$$

which contradicts the fact that $x \in \text{ext } \int_A F(t)d|m|$. Consequently, $x \in \int_A (\text{ext } F)(t)d|m|$, and so,

$$\text{ext } \int_A F(t)d|m| \subset \int_A (\text{ext } F)(t)d|m|.$$

(4) By theorem 4.2(1), $\int_A (\text{ext } F)(t)d|m|$ is convex. From propositions 3.10 and 3.12 and the results obtained above, we now have that

$$\int_A F(t)d|m| = \text{co ext } \int_A F(t)d|m| \subset \text{co } \int_A (\text{ext } F)(t)d|m| = \int_A (\text{ext } F)(t)d|m|.$$

Since

$$\int_A (\text{ext } F)(t)d|m| \subset \int_A F(t)d|m|,$$

we obtain the desired equality, namely

$$\int_A F(t)d|m| = \int_A (\text{ext } F)(t)d|m|. \qquad \nabla$$

4.4 <u>THEOREM</u> ([11], p. 263). <u>If</u> $m: \mathcal{C} \to V \subset L(U,W)$ <u>has the direct sum property and</u> Z <u>is a norming subspace of</u> W', <u>then there exists a function</u> $U_m: T \to L(U,Z')$ <u>having, among others, the following properties:</u>

(1) $\|U_m(t)\| = 1$ $|m|$-a.e. <u>on</u> T;

(2) $< U_m f, z >$ is $|m|$-integrable, and

$$< \int f(t)dm, z > = \int < U_m(t) f(t), z > d|m|,$$

for $f \in \mathcal{L}^1_U(m)$ and $z \in Z$;

(3) We can choose $U_m(t) \in L(U,W)$ for every $t \in T$ in the case that $W = Z'$.

4.5 REMARKS. (a) In the proof of theorem 4.4, the function U_m is defined in such a way that for every $u \in U$ and for every $z \in Z$, the function $\phi_{u,z}: T \to \mathbb{R}$, defined by $\phi_{u,z}(t) = < U_m(t)u, z >$, is locally $|m|$-integrable, that is, $\phi_{u,z}\chi_A$ is $|m|$-integrable for every set $A \in C$, see [11], p. 163, definition 1. According to 2.3.1(a), $\phi_{u,z}\chi_A$ is then $|m|$-measurable for every set $A \in C$. By [11], p. 100, corollary, $\phi_{u,z}\chi_A$ is $|m|$-measurable.
(b) Suppose now that $W = Z'$. Then, by theorem 4.4(3), we have that $U_m: T \to L(U,W)$. Definition 2.5.3 and (a) above then show that U_m is Z-weakly $|m|$-measurable. Suppose further that Z', and hence W, is separable. Then U_m is simply $|m|$-measurable, see [11], p. 105, proposition 22. If now $f: T \to U$ is $|m|$-measurable, then the function $g: T \to Z' = W$, defined by $g(t) = U_m(t)f(t)$, is $|m|$-measurable, see [11], p. 102, proposition 16. By theorem 4.4(1), we now have that

$$\| U_m(t) f(t) \| \leq \| U_m(t) \| . \| f(t) \| = \| f(t) \| \quad |m|\text{-a.e. on } T.$$

If $f \in \mathcal{L}^1_U(m)$, then $U_m f \in \mathcal{L}^1_W(|m|)$ by 2.3.1(d),(e). Under the conditions sketched above and from theorem 4.4(2) we obtain, for $f \in \mathcal{L}^1_U(m)$ and every $z \in Z$, that

$$< \int f(t)dm, z > = \int < U_m(t) f(t), z > d|m|$$
$$= < \int U_m(t) f(t) d|m|, z >.$$

The second equality above follows from [11], p. 123, corollary to the proposition 7. We then have that

$$\int f(t)dm = \int U_m(t) f(t) d|m|.$$

4.6 THEOREM. Let T be a countable union of sets of the ring C, U separable and $F: T \to U$ integrably bounded, point-compact and $|m|$-measurable. If W is separable, $W = Z'$ where Z is a norming subspace of W', $m: C \to V \subset L(U,W)$ and $U_m: T \to L(U,Z') = L(U,W)$ is the function whose existence is guaranteed by theorem 4.4, then

$$\int_A F(t)dm = \int_A U_m(t)F(t)d|m|, \quad \text{for every } A \in P(|m|).$$

PROOF. We observe first that the assumption on T, together with 2.1.2(f) implies that m has the direct sum property. Consequently, the function $U_m: T \to L(U,W)$ satisfying the properties of theorem 4.4 exists. Let $k \in \mathcal{L}^1_{\mathbb{R}}(|m|)$ be the bounding function. Consider the multifunction $U_mF: T \to W$, where

$$U_m(t)F(t) = \{U_m(t)u \mid u \in F(t)\}, \text{ for every } t \in T.$$

It is clear that U_mF is point-compact and that $U_m(t)F(t) \neq \emptyset$ for every $t \in T$. If for an arbitrary $t \in T$, $w \in U_m(t)F(t)$ then there exists an element $u \in F(t)$ such that $w = U_m(t)u$. Hence,

$$\|w\| = \|U_m(t)u\| \leq \|U_m(t)\|.\|u\| = \|u\| \leq k(t) \quad |m|\text{-a.e. on } T.$$

This shows that U_mF is also integrably bounded by k. We now show that U_mF is $|m|$-measurable. Since U is separable by assumption, we may apply proposition 3.3 and deduce that there exists a countable set $M = \{f_i \mid i \in I\}$ of $|m|$-measurable selectors of F such that, for every $t \in T$, $F(t) = \overline{M(t)}$. Since W is separable and $W = Z'$, we obtain from remark 4.5(b) that every function $g_i = U_mf_i$ is $|m|$-measurable. It follows from the definition of U_mF that every such g_i is an $|m|$-measurable selector of U_mF. For every $t \in T$, we have that $U_m(t)\overline{M(t)} \subset \overline{U_m(t)M(t)}$. The converse inclusion follows from:

$$U_m(t)M(t) \subset U_m(t)F(t) \Rightarrow \overline{U_m(t)M(t)} \subset \overline{U_m(t)F(t)}$$
$$= U_m(t)F(t) = U_m(t)\overline{M(t)}.$$

Combining these two inclusions, we have that the countable set $M_m = U_mM = \{U_mf_i \mid i \in I\}$ of $|m|$-measurable selectors of U_mF has the property that

$$\overline{M_m(t)} = \overline{U_m(t)M(t)} = U_m(t)\overline{M(t)} = U_mF(t), \text{ for every } t \in T.$$

The space W is separable. From the equivalent conditions (1) and (4) of proposition 3.3, applied to $U_mF: T \to W$, we see that

$\mathcal{U}_m F$ is $|m|$-measurable. Let $f \in \mathcal{S}_F$. Then $\mathcal{U}_m f$ is $|m|$-measurable by remark 4.5(b) and so $\mathcal{U}_m f \in \mathcal{S}_{\mathcal{U}_m F}$. Conversely, let $g \in \mathcal{S}_{\mathcal{U}_m F}$. We show that there exists a function $h \in \mathcal{S}_F$ such that $g(t) = \mathcal{U}_m(t)h(t)$ $|m|$-a.e. on T. Define the multifunction $G: T \to U$ by the equality

$$G(t) = \{u \in F(t) \mid \mathcal{U}_m(t)u = g(t)\}, \text{ for every } t \in T.$$

Then G is point-closed. In order to show that G is $|m|$-measurable, consider the function $k: T \times U \to W$ defined by $k(t,u) = \mathcal{U}_m(t)u$, for every $(t,u) \in T \times U$. For every $t \in T$, the function $k(t,\cdot) = \mathcal{U}_m(t)(\cdot): U \to W$ is continuous on U. By remark 4.5(b), the function $\mathcal{U}_m$ is simply $|m|$-measurable. Consequently, for every $u \in U$, the function $k(\cdot,u) = \mathcal{U}_m(\cdot)u: T \to W$ is $|m|$-measurable. Since U and W are both separable, we may apply [7], p. 97, lemma 3.1 and its corollary to the function k and deduce that k is $\mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$-measurable. Since $g: T \to W$ is also $|m|$-measurable, we have that the set

$$\{(t,u) \in T \times U \mid k(t,u) = g(t)\} = \{(t,u) \in T \times U \mid \mathcal{U}_m(t)u = g(t)\}$$

is $\mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$-measurable. Proposition 3.3 asserts that $\mathcal{G}(F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$, and so,

$$\mathcal{G}(G) = \mathcal{G}(F) \cap \{(t,u) \in T \times U \mid \mathcal{U}_m(t)u = g(t)\} \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U).$$

Proposition 3.3 asserts that G is $|m|$-measurable and corollary 3.2 in turn that G has an $|m|$-measurable selector h, say. Since $h(t) \in G(t)$ $|m|$-a.e., it follows that $h(t) \in F(t)$ $|m|$-a.e. and so $\mathcal{U}_m(t)h(t) = g(t)$ $|m|$-a.e. Since F and $\mathcal{U}_m F$ are both integrably bounded, we have that $\mathcal{S}_F = \mathcal{I}_F$ and $\mathcal{S}_{\mathcal{U}_m F} = \mathcal{I}_{\mathcal{U}_m F}$. We show that $\int F(t)dm = \int \mathcal{U}_m(t)F(t)d|m|$. Let $x \in \int F(t)dm$. There exists a function $f \in \mathcal{I}_F$ such that $x = \int f(t)dm = \int \mathcal{U}_m(t)f(t)d|m| \in \int \mathcal{U}_m(t)F(t)d|m|$ by remark 4.5(b) and since $\mathcal{U}_m f \in \mathcal{I}_{\mathcal{U}_m F}$. Conversely, if $y \in \int \mathcal{U}_m(t)F(t)d|m|$ then there exists a function $g \in \mathcal{I}_{\mathcal{U}_m F}$ such that $y = \int g(t)d|m|$. By what has been shown above, there exists a function $h \in \mathcal{I}_F$ such that $\mathcal{U}_m(t)h(t) = g(t)$ $|m|$-

a.e. From 2.3.1(c) and remark 4.5(b) we have that $y = \int g(t)d|m| = \int U_m(t)h(t)d|m| = \int h(t)dm \in \int F(t)dm$. This shows that

$$\int F(t)dm = \int U_m(t)F(t)d|m|.$$

Let A be an arbitrary set in $P(|m|)$ and define $F_A\colon T \to U$ by

$$F_A(t) = \begin{cases} F(t) & \text{if} \quad t \in A \\ \{0\} & \text{if} \quad t \in T - A. \end{cases}$$

Then

$$U_m(t)F_A(t) = \begin{cases} U_m(t)F(t) & \text{if} \quad t \in A \\ \{0\} & \text{if} \quad t \in T - A. \end{cases}$$

Now, F_A is integrably bounded, point-compact and $F_A(t) \neq \emptyset$ for every $t \in T$. Also, F_A is $|m|$-measurable, as shown in the proof of theorem 4.3. We may repeat the above procedure to show that

$$\int F_A(t)dm = \int U_m(t)F_A(t)d|m|.$$

As established in the proof of theorem 4.3, we have the equalities

$$\int_A F(t)dm = \int F_A(t)dm \quad \text{and}$$

$$\int_A U_m(t)F(t)d|m| = \int U_m(t)F_A(t)d|m|.$$

We conclude that

$$\int_A F(t)dm = \int_A U_m(t)F(t)d|m|$$

and this completes the proof. ∇

We now extend theorem 4.3 to the case that the integration is performed with respect to the vector measure m.

Suppose that $U = \mathbb{R}^n$, $V = \mathbb{R}^p$, $W = \mathbb{R}^{np}$, $f \in \mathcal{L}^1_{\mathbb{R}^n}(m)$ and $m\colon C \to \mathbb{R}^p$. We write

$$\int f(t)dm = (\int f(t)dm_1, \int f(t)dm_2, \ldots, \int f(t)dm_p) \in \mathbb{R}^{np},$$

where $m = (m_1, m_2, \ldots, m_p)$. We have that $\dim L(\mathbb{R}^n, \mathbb{R}^{np}) = n^2p$. We may consider $\mathbb{R}^p \subset L(\mathbb{R}^n, \mathbb{R}^{np})$ and the natural bilinear transformation $B\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{np}$, where $B(u,v) = v(u)$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^p \subset L(\mathbb{R}^n, \mathbb{R}^{np})$. We shall follow this line of thought in the theorem below.

4.7 THEOREM. Let T be a countable union of sets of the ring $\mathcal{C}$, $F: T \to \mathbb{R}^n$ integrably bounded, point-compact convex and $|m|$-measurable and let $m: \Sigma(|m|) \to \mathbb{R}^p$ be non-atomic. Then

$$\int_A F(t)dm = \int_A (\text{ext } F)(t)dm, \text{ for every } A \in \Sigma(|m|).$$

PROOF. (1) Put $Z = \mathbb{R}^{np}$; hence Z is a norming subspace of $(\mathbb{R}^{np})' = \mathbb{R}^{np}$. Consider $\mathbb{R}^p \subset L(\mathbb{R}^n, \mathbb{R}^{np})$; hence $m: \Sigma(|m|) \to \mathbb{R}^p \subset L(\mathbb{R}^n, \mathbb{R}^{np})$. It is clear that m has the direct sum property. Since all the conditions of theorem 4.4 are satisfied, there exists a function $U_m: T \to L(\mathbb{R}^n, Z') = L(\mathbb{R}^n, \mathbb{R}^{np})$ having all the properties mentioned in that theorem as well as in remarks 4.5. As was seen in the proof of theorem 4.6, the multifunction $U_m F: T \to \mathbb{R}^{np}$ is integrably bounded, point-compact, $|m|$-measurable and $U_m(t)F(t) \neq \emptyset$ for every $t \in T$. It also follows that $U_m F$ is point-convex, since F has this property. We now apply theorem 4.3 to $U_m F$ and we obtain

$$\int_A U_m(t)F(t)d|m| = \int_A (\text{ext } U_m F)(t)d|m|,$$

for every $A \in \Sigma(|m|)$.

This equality combined with the equality obtained in theorem 4.6 shows that

$$\int_A F(t)dm = \int_A U_m(t)F(t)d|m| = \int_A (\text{ext } U_m F)(t)d|m|,$$

where $A \in \Sigma(|m|)$.

(2) In order to show that $\int_A F(t)dm \subset \int_A (\text{ext } F)(t)dm$ for an arbitrary $A \in \Sigma(|m|)$, it suffices to show that $\int_A (\text{ext } U_m F)(t)d|m| \subset \int_A (\text{ext } F)(t)dm$ for such A. We may suppose that $|m|(A) > 0$. Now, $\mathcal{I}_{\text{ext } F} = \mathcal{S}_{\text{ext } F} \neq \emptyset$ and $\mathcal{I}_{\text{ext } U_m F} = \mathcal{S}_{\text{ext } U_m F} \neq \emptyset$ by the integrably boundedness of ext F and ext $U_m F$, respectively, and by proposition 3.7.

(3) Let $f \in \mathcal{I}_{\text{ext } U_m F}$. We show that there exists a function $h \in \mathcal{I}_{\text{ext } F}$ such that $U_m(t)h(t) = f(t)$ $|m|$-a.e. on T. Now, $f \in \mathcal{I}_{\text{ext } U_m F}$ implies that $f(t) \in (\text{ext } U_m F)(t)$ for every $t \in T - N$, where $N \subset T$ and $|m|(N) = 0$. This means that $\{f(t)\}$ is a closed extreme subset of $U_m(t)F(t)$ for $t \in T - N$. Define the multifunction $G: T \to \mathbb{R}^n$ by the equality

$$G(t) = \{u \in F(t) \mid \mathcal{U}_m(t)u = f(t)\}, \text{ for every } t \in T.$$

It is clear that G is point-closed. For every $t \in T - N$ we have that $G(t) \neq \emptyset$, $G(t)$ is an extreme subset of $F(t)$ by proposition 3.11 and $G(t) \cap (\text{ext } F)(t) \neq \emptyset$ by proposition 3.12. To see that $\mathcal{G}(G) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_{\mathbb{R}^n})$, we refer to the proof of theorem 4.6 and the definition of the multifunction G in that theorem. Proposition 3.8 asserts that $\mathcal{G}(\text{ext } F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_{\mathbb{R}^n})$. Consider the multifunction $H: T \to \mathbb{R}^n$, defined by the equality

$$H(t) = G(t) \cap (\text{ext } F)(t), \text{ for every } t \in T.$$

Then,

$$\mathcal{G}(H) = \mathcal{G}(G) \cap \mathcal{G}(\text{ext } F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_{\mathbb{R}^n})$$

by proposition 3.9. Also, $H(t) \neq \emptyset$ for every $t \in T - N$. Put $T_1 = T - N$ and $H_1 = H|T_1$. Consider the σ-algebra $\mathcal{P}(T_1, |m|)$ of all $|m|$-measurable subsets of T_1. We now have that

$$\mathcal{G}(H_1) \in \mathcal{T}(\mathcal{P}(T_1, |m|) \times \mathcal{B}_{\mathbb{R}^n}) \subset \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_{\mathbb{R}^n}).$$

It is clear that $H_1(t) \neq \emptyset$ for every $t \in T_1$ and that the measure space $(T_1, \mathcal{P}(T_1, |m|), |m|^*)$ is σ-finite and complete. An application of proposition 3.4 leads to the existence of a $\mathcal{P}(T_1, |m|)$-measurable function $h_1: T_1 \to \mathbb{R}^n$ such that $h_1(t) \in H_1(t)$ for every $t \in T_1$. Let x_o be an arbitrary element of $\mathbb{R}^n$ and define the function $h: T \to \mathbb{R}^n$ by

$$h(t) = \begin{cases} h_1(t) & \text{if} \quad t \in T_1 \\ x_o & \text{if} \quad t \in N. \end{cases}$$

Then $h(t) \in H(t)$ for every $t \in T_1$ and h is clearly $|m|$-measurable. It follows that $h \in \mathcal{S}_{\text{ext } F} = \mathcal{I}_{\text{ext } F}$ and that $\mathcal{U}_m(t)h(t) = f(t)$ $|m|$-a.e. on T.

(4) Suppose now that $x \in \int_A (\text{ext } \mathcal{U}_m F)(t)d|m|$. Then there exists a function $f \in \mathcal{I}_{\text{ext } \mathcal{U}_m F}$ such that $x = \int_A f(t)d|m|$. By what has been shown in part (3) above, there exists a function $h \in \mathcal{I}_{\text{ext } F}$ such that $\mathcal{U}_m(t)h(t) = f(t)$ $|m|$-a.e. on T. Consequently,

$$\begin{aligned} x = \int_A f(t)d|m| &= \int_A \mathcal{U}_m(t)h(t)d|m| \\ &= \int \mathcal{U}_m(t)h(t)\chi_A(t)d|m| \end{aligned}$$

$$= \int h(t)\chi_A(t)\,dm$$

$$= \int_A h(t)\,dm \in \int_A (\text{ext } F)(t)\,dm$$

by using the observations made in remark 4.5(b). This shows that

$$\int_A F(t)\,dm = \int_A (\text{ext } U_m F)(t)\,d|m| \subset \int_A (\text{ext } F)(t)\,dm.$$

Since the converse inclusion

$$\int_A (\text{ext } F)(t)\,dm \subset \int_A F(t)\,dm$$

obviously holds, we now have the required equality

$$\int_A F(t)\,dm = \int_A (\text{ext } F)(t)\,dm. \qquad \nabla$$

4.8 REMARKS. Theorem 4.7 has also been proved by:
(a) A. Dvoretzky, A. Wald and J. Wolfowitz [13] for the case that $F: T \to \mathbb{R}^{n+1}$ is such that $F(t) = \Lambda$ for every $t \in T$, where Λ is an n-dimensional simplex; (b) S. Karlin and W.J. Studden [17] for the case that $F: T \to \mathbb{R}^n$ is such that $F(t) = C$ for every $t \in T$, where C is a fixed compact convex subset of $\mathbb{R}^n$;
(c) C. Castaing [8] for the case that T is a compact metric space and $F: T \to X \subset \mathbb{R}^n$ point-compact convex where X is a non-empty compact convex metrizable subset of $\mathbb{R}^n$, and
(d) M. Valadier [28] for the case that F is scalarwise integrable (see 2.4), and $|m| = |m_1| + |m_2| + \ldots + |m_p|$ where $m = (m_1, m_2, \ldots, m_p)$. In each of the cases (a)-(d), the measure with respect to which the integration is performed is vector-valued and non-atomic.

5. EXAMPLES

The main purpose of this section is to show by means of illustrative examples that parts of the hypotheses of theorems 4.3 and 4.7 cannot be weakened.

5.1 EXAMPLE. Let $T = \{t_o\}$, $\Sigma = \{\emptyset, T\}$ and $m: \Sigma \to \mathbb{R}$ be defined by $m(T) = 1$, $m(\emptyset) = 0$. Then m is an atomic measure and $m = |m|$. Define $F: T \to \mathbb{R}$ by $F(t) = [\,1,2\,]$. Then F satisfies the conditions of theorems 4.3 and 4.7. Furthermore,

$$(\text{ext } F)(t) = \{1,2\} = \int (\text{ext } F)(t)dm.$$

If $f: T \to \mathbb{R}$ is defined by $f(t) = 1\frac{1}{2}$, then $f \in I_F$ and $\int f(t)dm = 1\frac{1}{2} \in \int F(t)dm$. Thus,

$$\int F(t)dm \neq \int (\text{ext } F)(t)dm.$$

5.2 EXAMPLES. Let $T = [0,1]$, Σ be the Lebesgue σ-algebra of subsets of T and m the Lebesgue measure on T. Then m is non-atomic and $m = |m|$. (a) Define $F: T \to \mathbb{R}$ by $F(t) = \mathbb{R}$ for all $t \in T$. Then F is point-convex, but neither integrably bounded nor point-compact. Clearly, $(\text{ext } F)(t) = \emptyset$ for all $t \in T$ and

$$\int (\text{ext } F)(t)dm = \emptyset \neq \int F(t)dm.$$

(b) Define $F: T \to \mathbb{R}$ by $F(t) = (0,1)$ for all $t \in T$. Then F is integrably bounded and point-convex but not point-compact. As in (a) above we have that

$$\int (\text{ext } F)(t)dm = \emptyset \neq \int F(t)dm.$$

5.3 EXAMPLE. The space c_o is the Banach space of all sequences $x = (x_n)$ converging to zero. The space c_o is infinite dimensional and the closed unit ball A of c_o is non-compact and convex. Let T, Σ and m be as in 5.2 and consider $c_o = L(\mathbb{R}, c_o)$. Define $F: T \to c_o$ by $F(t) = A$ for all $t \in T$. Then F is clearly $|m|$-measurable and integrably bounded. Since $\text{ext } A = \emptyset$ (see [14], p. 709), we have that

$$\int (\text{ext } F)(t)dm = \emptyset \neq \int F(t)dm.$$

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