CONCERNING EXTENDABLE CONNECTIVITY FUNCTIONS

By Jerry Gibson

In the classic paper [16], J. Stallings asked the following question: "If $I = [0,1]$ is embedded in I^2 as $I \times \{0\}$, can a connectivity function $f:I \rightarrow I$ be extended to a connectivity function $g:I^2 \rightarrow I$?" Negative answers were given to this question by Cornette [4] and Roberts [14]. Each constructed a connectivity function $I \rightarrow I$ that is not an almost continuous function.

Definition 1. $f:X \rightarrow Y$ is a connectivity function if and only if the graph of f restricted to C is connected in $X \times Y$ whenever C is connected in X.

Definition 2. $f:X \rightarrow Y$ is an almost continuous function if and only if each open subset of $X \times Y$ containing the graph of f contains the graph of a continuous function with the same domain.

In this paper all propositions will be restricted to I, I^2, or $I \times \{p\}$ where $p \in I$ even though they may have been proved (or maybe proved) more generally.

Proposition 1. If $f:I \rightarrow I$ is an almost continuous function, then f is a connectivity function, [16].

Proposition 2. If $f:I^2 \rightarrow I$ is a connectivity function, then f is an almost continuous function, [16].

Proposition 3. $f:I \rightarrow I$ is a connectivity function if and only if the entire graph of f is connected.

Proposition 4. If $f:I^2 \rightarrow I$ is a connectivity (or almost continuous) function, then $f \mid (I \times \{0\})$ is a connectivity (or almost continuous) function, [16].

Because of these propositions, Cornette and Roberts were able to give a negative answer to the question posed by Stallings. In [13], K. R. Kellum proved that an almost continuous function $f:I \rightarrow I$ can be extended to an almost continuous function $g:I^2 \rightarrow I$. Thus a natural question arises.

Question 0. Can an almost continuous function $f:I \rightarrow I$ be extended to a connectivity function $g:I^2 \rightarrow I$?

In what follows we give a negative answer to this question and prove related results.

Stallings [16] stated that if $\pi_x:I^2 \rightarrow I \times \{0\}$ is the projection and $f:I \rightarrow I$ is a connectivity function, then $f \circ \pi_x :I^2 \rightarrow I$ may not be a connectivity function. In fact we have the following proposition.

Proposition 5. If $g:I^2 \rightarrow I$ is continuous and onto and $f:I \rightarrow I$ is any function such that $f \circ g:I^2 \rightarrow I$ is a connectivity function, then f is continuous except perhaps at 0 or 1, [7].

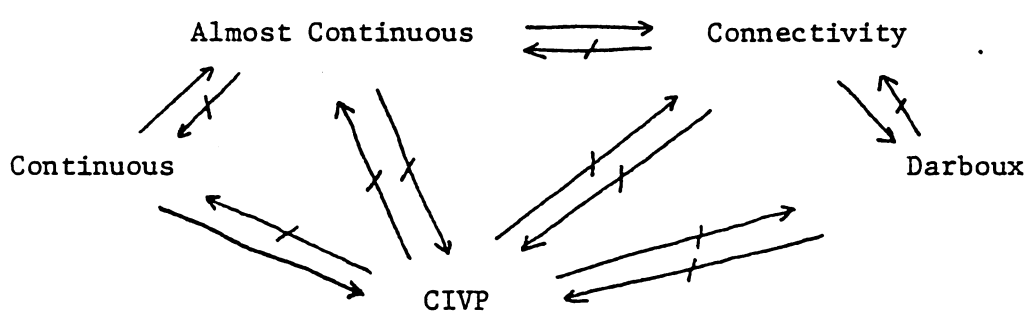
A related function is defined as follows:

Definition 3. $f:X \rightarrow Y$ is said to be peripherally continuous at x if and only if for each open subset U of X containing x and for each open subset V or Y containing f(x) there exists and open subset W of U containing x such that f(bd(W)) is a subset of V.

Proposition 6. $f: I^2 \to I$ is a connectivity function if and only if f is peripherally continuous, [10].

Definition 4. $f: I \to I$ has the Cantor Intermediate Value Property (CIVP) if and only if for each $a, b \in I$ such that $f(a) \neq f(b)$ and for any Cantor set K between f(a) and f(b) there exists a Cantor set C between a and b such that $f(C) \subset K$, [5].

The following diagram was completed in [5]:



The first example of that paper is an almost continuous function $f: I \to I$ that does not have the CIVP.

Proposition 7. If $f: I \to I$ is closed and has the CIVP, then f is continuous, [5].

Definition 5. $f: I \to I$ has the weak CIVP (WCIVP) if and only if for any $a, b \in I$ such that $f(a) \neq f(b)$ there exists a Cantor set C between a and b such that f(C) is between f(a) and f(b).

The first example of [5] is an example of an almost continuous function $f: I \to I$ that does not have the WCIVP.

Proposition 8. If $g:I^2 \to I$ is an extension of $f:I \to I$ and g is a connectivity function, then f has the WCIVP. Moreover, the Cantor set can be selected so that f restricted to it is continuous, [6].

Thus it follows that the first example of [5] is an almost continuous function $f:I \to I$ that can not be extended to a connectivity function $g:I^2 \to I$. This gives a negative answer to Question 0.

Definition 6. A function f: $[a,b] \to$ Reals is said to have a perfect road at $x \in [a,b]$ provided that

(1) there exists a perfect set P such that x is a bilateral point of accumulation of P and

(2) f restricted to P is continuous at x.

In each case, when we deal with the endpoints of [a,b], the bilateral condition is replaced with a unilateral condition.

Proposition 9. Let $g:I^2 \to I$ be an extension of $f:I \to I$. If g is a connectivity function, then f has a perfect road at each point. Moreover, f restricted to this perfect set is continuous at each of its points, [8].

The proof of the following proposition follows in a similar way as the proof of proposition 8.

Proposition 10. If $g:I^2 \to I$ is a connectivity function, then $f = g \mid (I \times \{x\})$, for any $x \in I$, has the following property(gr): If $[a,b] \subset I$, then there exists a Cantor set $C \subset (a,b)$ such that $f \mid C$ is continuous where I is embedded in I^2 as $I \times \{x\}$.

In a similar manner we have the following proposition:

Proposition 11. If $g:I^2 \to I$ is a connectivity function, then $f = g \mid (I \times \{x\})$ has a perfect road P at each point for any $x \in I$. Moreover, the perfect set P can be selected so that $f \mid P$ is continuous at each point of P.

However, there exists connectivity functions $I \to I$ that have neither of these properties.

Definition 7. A function $f:X \to Y$ is said to have property(s) provided that if $P \subset X$ is a perfect set, then there exists a perfect subset $Q \subset P$ such that $f \mid Q$ is continuous

This property was studied by Sierpinski [15] and Szpilranjn-Marczewski [17]. Property (gr) is a special case of property(s).

It follows that there exists a connectivity function $g:I^2 \to I$ and there exists $p \in I$ such that $f = g \mid (I \times \{p\})$ does not have property(s). However, it does have the property(gr). This example has not yet been submitted for publication.

We now state the following questions:

Question 1. Does there exist an almost continuous function $f:I \to I$ that has a perfect road at each point but can not be extended to a connectivity function $g:I^2 \to I$?

Question 2. Does there exist a Baire class 1 connectivity function $f:I \to I$ that can not be extended to a connectivity function $g:I^2 \to I$?

Question 3. If $g:I^2 \to I$ is a connectivity function and $f:I \to I$ is a function such that $f \circ g:I^2 \to I$ is a connectivity function, is f continuous except perhaps at 0 or 1?

Question 4. If $g:I^2 \to I$ is an extension of $f:I \to I$ and g is a connectivity function, does f have the CIVP?

Question 5. Is it true that if $f:I \to I$ can be extended to a connectivity function $g:I^2 \to I$, then f can be extended to a connectivity function $g:I^2 \to I$ such that g is continuous on the complement of $I \times \{0\}$?

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