Determining Sets for Functions and Measures

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This article abstracts the contents of two talks delivered at the Ninth Annual Real Analysis Symposium in Louisville on June 13, 1985, under the title "The Uses of Analyticity." The lectures addressed the following question: given a function or a measure on some appropriate space (such as R^n), to what extent is it determined by its integrals over certain geometrically pleasant subsets (lines, planes, balls, etc.) of the space in question? This is a subject with an extensive literature. In an attempt to make the discussion manageable (and in line with my own research interests), the talks focused on aspects of the problem in which harmonic analysis and complex function theory play a central role. Here an amusing personal coincidence may be mentioned. My interest in the subject was first aroused by reading a short paper [3] by one of the giants of twentieth century real analysis, Abram Samoilovich Besicovitch. (The problem considered in [3] turned out to have been solved years earlier, but that is where I first saw it.) Now it just so happens that I first learned complex variables from Besicovitch (at Dartmouth College in 1962-63). Thus Besicovitch is, in a certain sense, doubly responsible for my activity in this area; and I think it appropriate to dedicate this paper to his memory.

We shall find it convenient to adopt vector notation. Thus, $x = (x_1 x_2, \ldots, x_n)$, $\quad = (\ _1, \ _2, \ldots, \ _n)$, etc. In much of the following, $n = 2$.

1. Let f be defined on R^2 and suppose that

$$(*) \qquad \int_{\ell} f\, ds = 0$$

for each line ℓ. Must f vanish almost everywhere?

When $f \in L'$, it is well-known that the answer is yes. The simplest proof of this fact [18] proceeds by introducing the Fourier transform

$$f(\xi) = \cdot \quad e^{-i(x- \ )} f(x)\,dx$$

of f and observing (via Fubini's theorem and the uniqueness theorem for the one-dimensional Fourier transform) that f vanishes on a line $\tilde{\ell}$ through the origin if and only if f satisfies (*) for almost all lines perpendicular to $\tilde{\ell}$. In particular, if (*) holds unrestrictedly, f must vanish identically; and $f = 0$ a.e. by Fourier uniqueness. Actually, since $\hat{f}$ is a continuous function on R^2, it suffices for (*) to hold for almost every line belonging to an arbitrary dense set of directions. Thus, *an integrable function f which satisfies (*) for almost every line in each of a dense set of directions must vanish almost everywhere*.

Without additional restrictions on f, no further improvement is possible. Indeed, let α be an arc of the unit circle and take D to be a disc contained in the angle subtended at the origin by α. Choosing a smooth function $\phi = 0$ supported in D and putting $f = \hat{\phi}$, we have, by Fourier inversion,

$$\hat{f}(\xi) = \hat{\hat{\phi}}(\xi) = (2\pi)^2 \ (-\xi) = 0$$

on any line through 0 whose direction does not belong to α. It follows that the integral of f vanishes along every line perpendicular to any direction not in α. Moreover, since ϕ belongs to the Schwartz class $\mathcal{S}$ of smooth, rapidly decreasing functions, so does f. Thus, *given any open set of directions H there exists a nonzero function in $\mathcal{S}$ which satisfies* (*) *for all lines whose directions are not in* H.

When f vanishes off a bounded set, the situation changes dramatically. In that case, $\hat{f}$ is real analytic on R^2; in fact, it extends to an entire function on $\mathbb{C}^2$. Such a function cannot vanish on an infinite collection of lines through 0 without vanishing identically. Thus, *if f is integrable and has compact support and* (*) *holds for almost all lines in each of an arbitrary infinite set of directions then* $f = 0$ *a.e.* It is easy enough to construct smooth functions of compact support whose Fourier transforms vanish on any finite collection of lines through 0; thus, the above result is sharp. On the other hand, the characteristic function of a convex set is (in general) determined by all integrals in four directions [7]. It would be interesting to have a proof of this fact based on Fourier analysis and complex function theory.

The hypothesis that f has compact support can be viewed as the stipulation that f(x) be small for large values of x; alternatively, it can be taken to say that the support of f has a large complement. Each interpretation suggests a natural generalization. In the first instance, suppose there exist constants k, c 0 such that

$$|f(x_1,x_2)| \leq k\ e^{-c(|x_1|+|x_2|)}.$$

Then $\hat{f}$ will extend to a function holomorphic in a slab about the real subspace $\mathbb{R}^2 \subset \mathbb{C}^2$, hence real analytic on R^2; and the result of the previous paragraph again obtains.

For a generalization of the second sort, suppose f is supported in the first quadrant $\{(x_1,x_2): x_1 \geq 0,\ x_2 \geq 0\}$. Then

$$\hat{f}(\zeta_1,\zeta_2) = \qquad e^{-i(x_1\zeta_1+x_2\zeta_2)} f(x_1,x_2)\,dx_1 dx_2$$

defines a continuous function bounded on $\{(\zeta_1,\zeta_2): \operatorname{Im}\zeta_1 \leq 0,\ \operatorname{Im}\zeta_2 \leq 0\}$ and analytic on its interior. Since $\mathbb{R}^2$ is the distinguished boundary of the bidisc $\{\operatorname{Im}\zeta_1 < 0\} \times \{\operatorname{Im}\zeta_2 < 0\}$, the boundary values of $f(\zeta_1,\zeta_2)$ -- which agree with the ordinary Fourier transform of f -- cannot vanish on a set of positive measure on $\mathbb{R}^2$ without vanishing identically. Thus, $\hat{f}$ cannot vanish on a set of lines through 0 whose directions form a set of positive (linear) measures unless $f = o$ a.e. It is not difficult to adapt this argument to functions whose supports lie in an angle of opening less than π. In summary, <u>if the support of the integrable function</u> f <u>lies in an angle of opening less than</u> π <u>and</u> (*) <u>holds for almost every line in a set of directions having positive measure, then</u> $f = 0$ a.e.

The argument given above fails when applied to functions whose support is a half-plane. In that case, the Fourier transform can actually vanish on an open set. Indeed, put $f(x_1,x_2) = g(x_1)h(x_2)$, where the integrable function g satisfies $g(x_1) = 0$ for $x_1 < 0$ and $h = \hat{\phi}$, $\hat{\phi}$ a smooth function of compact support. Clearly, f is supported on the half plane $x_1 \geq 0$. But

$$\hat{f}(\xi_1,\xi_2) = \hat{g}(\xi_1)\hat{h}(\xi_2) = \hat{g}(\xi_1)\hat{\hat{\alpha}}(\xi_2) = 2\pi\hat{g}(\xi_1)\alpha(-\xi_2),$$

which vanishes off some horizontal strip. Nonetheless, the *result* of the previous paragraph remains valid. Since I cannot recall having seen this fact noted previously in print, let me give an indication of the proof. If $f(x_1,x_2) = 0$ for $x_1 < 0$, the integral

$$\hat{f}(\zeta_1,\xi_2) = \quad e^{-i(x_1\zeta_1+x_2\xi_2)} f(x_1,x_2)dx_1dx_2$$

defines a bounded analytic function of ζ_1 for $\zeta_1 < 0$. Thus, for fixed ξ_2, $g(\xi_1) = \hat{f}(\xi_1,\xi_2)$ is the boundary value function of a bounded analytic function on a half-plane and thus cannot vanish on a set of positive (linear) measure without vanishing identically. If $\hat{f}$ vanishes on a set of lines through 0 whose directions have positive linear measure, its restriction to any horizontal line in the plane other than the x -axis also vanishes on a set of positive measure; hence $\hat{f} = 0$ and $f = 0$. It follows that *if the support of the integrable function f lies in a half-plane and (*) holds for almost every line in a set of directions having positive measure, then* $f = 0$ a.e.

Throughout our discussion it has been assumed that $f \in L'$. Without some assumption of measurability, there is no possibility of concluding that $f = 0$ a.e., even if (*) holds as an absolutely convergent integral for all ℓ. Indeed, according to a celebrated example of Sierpinski [12], there exists a nonmeasurable set E whose intersection with each line in the plane consists of at most two points. The characteristic function $f = \chi_E$ then satisfies (*) for all ℓ but does not vanish a.e. If a function of compact support with the same property is desired, it suffices to consider $\chi_{E\ D}$, where D is a sufficiently large disc about the origin.

Even if f is measurable, it can satisfy (*) for almost every line in each direction without vanishing identically. Indeed, putting $z = x_1 + ix_2$ and setting $f(x_1,x_2) = 1/z^k$ $(k \geq 2)$, we see that (*) holds for any line which does not pass through 0. Of course, this f fails to be locally integrable near 0.

Surprisingly, the question of whether a locally integrable function can satisfy (*) for <u>all</u> lines without vanishing identically does not seem to have been investigated until relatively recently. In [18], I constructed an example of an <u>entire</u> function with this property. This example may be viewed as an elaboration of the example of the previous paragraph in which the pole at 0 is pushed out to infinity. Here is a brief sketch of the construction. Let $z = x_1 + ix_2$ and put

$$D = \{z : |z| < 3\} \qquad S = \{x_1 + x_2 : \frac{1}{x_1} < x_2 < \frac{2}{x_1}, x_1 > 1\}$$

$$U = D \cup S \qquad K = \mathbb{C}/U$$

An important theorem in approximation theory due to N. U. Arakelyan allows us to construct an entire function $g(z)$ such that

$$\left|\frac{1}{z^3} - g(z)\right| < \frac{1}{|z^3|} \qquad z \in K$$

Clearly g is nonconstant; moreover, for $z \in K$ we have $|g(z)| \leq 2/|z|^3$. A standard estimate using Cauchy's integral formula for derivatives gives $g'(z) = O(|z|^{-2})$ along each line; hence g' is absolutely integrable on every line. Set $f = g'$ and fix ℓ. Then $ds = c\,dz$ on ℓ, so by the fundamental theorem of calculus and the fact that $g(z) \to 0$

as $z \to \infty$ on ℓ it follows that the integral of f over ℓ vanishes. Thus (*) holds for all lines.

To conclude this section, we pose the natural question of extending the previous example to higher dimensions. Fix n and k $(0 < k < n)$ and suppose

$$(**) \qquad \int_F f \, dx = 0$$

for every affine k-flat F in R^n. If f $L'(R^n)$, it follows as before that $f = 0$ a.e. What if $f \in C(R^n)$ and (**) holds as an absolutely convergent integral for each F, but no assumption of global integrability is made? A straightforward adaptation of the construction just given would require proving theorems about tangential approximation by harmonic functions in space analogous to the results of Arakelyan used above. Recently, Shaginyan announced some results in this direction [11]; but the degree of approximation he obtains is not adequate for the applications envisaged above even in the case $n = 3$, $k = 1$. On the other hand, if $f \in C(R^n)$ and there exist <u>two</u> integers k and m, $0 < k < m < n$, such that (**) holds in the sense of absolute convergence for all k-flats and all m-flats, it follows that $f = 0$. The simple proof of this result is left as an exercise for the reader.

2. According to a well-known result of Cramer and Wold [5], a probability measure on $\mathbb{R}^n$ is determined by its values on all half-spaces. More generally, if μ is a finite complex Borel measure and $\mu(H) = 0$ for every half-space then $\mu = 0$. For absolutely continuous measures, this is easily seen to be equivalent to the result discussed at the very beginning of the previous lecture. The general case follows in routine fashion (or can be derived directly from essentially the same argument used to handle the

absolutely continuous case). In this section, we shall be concerned with the consequences of replacing the condition $\mu(H) = 0$ by $\mu(B) = 0$, where B ranges over a specified collection of balls. In this case, there is obviously no need for any global finiteness conditions on μ; and, indeed, many of the most interesting results concern the case of measures whose total variation is infinite.

The question to be considered makes sense, and is interesting, in a fairly general context. Let X be a metric space, μ a measure on X. Suppose $\mu(B) = 0$ for all closed balls in X. Must $\mu = 0$? In general, the answer is no. Indeed, R.O. Davies has constructed a compact metric space X supporting distinct probability measures μ_1 and μ_2 such that $\mu_1(B) = \mu_2(B)$ for every closed ball in X [6].

Predictably, Davies' example is highly nonhomogeneous. In those situations in which the space X exhibits a certain degree of homogeneity, an affirmative answer can be obtained. For instance, suppose the (locally compact metric) space X supports a uniform measure. This means that there exists a positive measure m on X such that $m(B(x,r))$ is independent of x and $0 < m(B(x,r)) < \infty$ for all $x \in X$, $0 < r < \infty$. Here, as usual, $B(x,r)$ denotes the closed ball of radius r about the point $x \in X$. Then if the (signed or complex) measure μ satisfies $\mu(B) = 0$ for all closed balls B, it follows that $\mu = 0$ [4]. Here is a sketch of the proof. Set

$$K_\varepsilon(x,y) = \begin{cases} \dfrac{1}{m(B(x,\varepsilon))} & d(x,y) < \varepsilon \\ 0 & d(x,y) \geq \varepsilon \end{cases}$$

and put

$$(K_\varepsilon \phi)(x) = \int_X K_\varepsilon(x,y)\phi(y)\ dm(y)$$

for ϕ continuous and of compact support. One sees easily that

$$\int_X (K_\varepsilon \phi)(x)\ d\mu(x) \to \int_X \phi(x) d\mu$$

as $\varepsilon \to 0$. On the other hand, by Fubini's theorem,

$$\int_X (K_\varepsilon \phi)(x)\ d\mu(x) = 0$$

for all ε. Hence $\int \alpha d\mu = 0$; and so $\mu = 0$, since μ is arbitrary.

It can also be shown that if X is finite dimensional (in the sense that there exists an integer n such that every ball in X can be covered by n balls of half its radius) then any measure which vanishes on all balls must vanish identically. There is also a large class of (infinite dimensional) Banach spaces for which this holds true, including Hilbert space, L^p spaces $(1 < p \leq \infty)$, $C(K)$, C_0, etc. See [8] for further information on this topic.

Since Lebesque measure is a uniform measure on R^n, any (complex Borel) measure on R^n is determined by its values on all balls. However, in the Euclidean setting, much more is true.

Fix $r > 0$ and let $B = B_r = B(o,r) = \{x \in R^n: |x| \leq r\}$. Then

$$\mu(B(x,r)) = \int_{B(x,r)} d\mu = \int \chi_{B(x,r)}(y)\,d\mu(y)$$
$$= \int \chi_B(y-x)\,d\mu(y)$$
$$= \int \chi_B(x-y)\,d\mu(y)$$
$$= \int \chi_B * \mu(x),$$

where χ_S denotes the characteristic function of the set S. Thus the condition $\mu(B(x,r)) = 0$ for all x is equivalent to the convolution equation

$$\chi_B * \mu = 0.$$

In case μ is a finite measure, we may take Fourier transforms to obtain the relation

$$\hat{\chi}_B(\xi)\hat{\mu}(\xi) = 0.$$

Now $\hat{\chi}_B$, being the transform of a function of compact support, is the restriction to R^2 of an entire function on C^2 and hence does not vanish on an open set. Since μ is finite, $\hat{\mu}$ is continuous on R^2. It follows that we must have $\hat{\mu} = 0$, hence (by Fourier uniqueness) $\mu = 0$. It should be noted that this argument has nothing to do with balls; it applies when B is replaced by <u>any</u> compact set of positive Lebesque measure.

In case $|\mu|(R^n) = \infty$, μ does not have a Fourier transform in the classical sense and the above argument fails. Is this simply a failure of method, or does some genuinely new phenomenon arise in the general case? The question is not idle. In fact, it has a connection with analytic function theory. One version of Morera's theorem says that if $f \in C'(D)$ and $\int_\Gamma f(z)dz = 0$ for every circle Γ in D, then f is holomorphic on D. Suppose $D = R^2$ and $\int_\Gamma f(z)dz = 0$ for each circle of fixed radius r_0. Must f be entire? The connection with our previous question is provided by Green's Theorem:

$$\int_\Gamma f(z)dz = 2i \iint_\Delta \frac{2f}{2\bar{z}} \, dxdy,$$

where Δ is the disc bounded by Γ. Now f will be analytic if $\frac{2f}{2z} = 0$, but there is obviously no warrant for assuming $\iint |\frac{2f}{2\bar{z}}| \, dxdy < \infty$ at the outset.

Let us return to our original question: for general (not necessarily finite) measures on R^n, does $\mu(B) = 0$ for all balls of (fixed) radius r imply that $\mu = 0$? The answer turns out to be negative. There are various ways to see this, but one of the nicest is via the formula of Pizzetti [16, p. 342]. To simplify notation, let us take $n = 2$, so that we are working in the plane. Pizzetti's formula then says

$$\int_{B(x,r)} u(t)dt = 2\pi r \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{k!(k+1)!} \left(\frac{r}{2}\right)^{2k+1},$$

for u real entire. Write $t = (\sigma, \tau)$ and put $u(t) = e^{i\tau}$. Then $\Delta u = -u$, $\Delta^k u = (-1)^k u$, whence

$$\int_{B(x,r)} u(t)dt = 2\pi r \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(kH)!} \left(\frac{r}{2}\right)^{2k+1} . u(x)$$

$$= 2\pi r . J_1(r) . u(x),$$

where J_1 is the Bessel function of order 1. It is well known that J_1 has an infinite number of positive real zeros $r_1, r_2, \ldots$ For each such r and all $x \in \mathbb{R}^2$ we have $\int_{B(x,r)} u\,dt = 0$.

In the opposite direction we have the following

<u>Theorem</u> [15]. Let μ be a complex Borel measure on R^n and suppose that $\mu(B) = 0$ for every ball having radius r_1 or r_2. Then $\mu = 0$ so long as r_1/r_2 is not equal to a quotient of zeros of the Bessel function $J_{n/2}(z)$.

This result is sharp in the sense that it no longer holds if r_1/r_2 <u>is</u> a quotient of zeroes of $J_{n/2}$; this is already evident from the example of the measure $\mu = udx$ constructed immediately above. It is also worth noting that the set of proscribed ratios forms a countable dense set. Thus, while almost any pair of numbers r_1, r_2 will give rise to an affirmative conclusion, a slight change in either r_1 or r_2 (or both) can lead from a positive result to no result at all. This inherent instability places a severe limitation on the usefulness of the theorem in applications. A fairly complete discussion of this and related results can be found in [17]. Thus, we content ourselves with the remark that the proof is an easy (if somewhat unexpected) application of the theory of mean periodic functions of one variable to the pair of convolution equations $\chi_{B_{r_1}} * \mu = 0$, $\chi_{B_{r_2}} * \mu = 0$.

Analogous results obtain in spaces of constant curvature. For instance, let $S(n,-\alpha^2)$ denote the (unique, up to isometric equivalence) complete, simply-connected, n-dimensional Riemannian manifold of constant curvature $-\alpha^2 < 0$. Suppose μ is a complex Borel measure on $S(n,-\alpha^2)$

and $\mu(B) = 0$ for all (geodesic) balls of radius r_1 and r_2. Then $\mu = 0$ so long as the equations

$$P_z^{-n/2}(\cosh \alpha r_1) = 0 \qquad P_z^{-n/2}(\cosh \alpha r_2) = 0$$

have no common solution $z \in \mathbb{C}$. Here $P_z^{-n/2}(x)$ is the associated Legendre function of the first kind. Again these results are sharp. It is amusing to observe [1, p. 122] [17, p. 170] that if we scale the preceding equations by an appropriate factor and let the curvature tend to zero, we obtain formally the condition that the equations

$$J_{n/2}(r,z) = 0 \qquad J_{n/2}(r_2 z) = 0$$

have no common nonzero solution. This is just the requirement that r_1/r_2 not be a quotient of zeros of $J_{n/2}$ of the theorem above (which corresponds to $\alpha = 0$).

The spaces of constant positive curvature $S(n,\alpha^2)$ are simply the ordinary spheres $S^n(1/\alpha)$ of radius $1/\alpha$ in R^{n+1}. In this case, all spaces are compact, all measures are necessarily finite, and the corresponding results have a somewhat simpler character. Let μ be a measure on $S^n(1/\alpha)$ such that $\mu(B) = 0$ for every geodesic ball (= spherical cap) of fixed radius r_1. Then $\mu = 0$ so long as r_1 is not a zero of any of the functions $C_m^{(n+1)/2}(\cos \alpha r)$ $m = 1,2,3,\ldots$ Here the $C_m^{(n+1)/2}$ are Gegenbauer polynomials. (This result can also be stated in terms of the associated Legendre polynomials $P_{m+n/2}^{-n/2}$.)

All these results extend to a much more general situation, viz. to the case of rank one symmetric spaces. For a detailed treatment, including a general overview of the subject and a discussion of the previous literature, see [2].

We conclude this paper with a brief discussion of the Pompeiu problem. A compact set $E \subset \mathbb{R}^n$ of positive Lebesque measure is said to have the Pompeiu property if, whenever $\mu(\sigma(E)) = 0$ for all rigid motions σ of R^n, it follows that $\mu = 0$. We have already observed that balls fail to have the Pompeiu property; in fact, the only compact sets which are known not to have the Pompeiu property arise in fairly simple fashion as differences of balls. Is the ball the only element of its homeomorphism class which fails to enjoy the Pompeiu property? This natural question has proved surprisingly refractory. It has an entertaining reformulation as a free boundary problem in PDE, which we state for the case $n = 2$.

Let E be a Jordan domain and suppose there exists a number $\lambda > 0$ (an "eigenvalue") such that the boundary value problem

$$(P) \qquad \begin{array}{ll} \Delta\mu + \lambda\mu = 0 & \text{on } E \\ \mu = c \quad \frac{\partial\mu}{\partial n} = 0 & \text{on } \partial E \end{array}$$

has a solution. Must E be a disc?

When $E = \{(x_1,x_2):x_1^2 + x_2^2 \leq R^2\}$, the function $u(x_1,x_2) = J_0(\sqrt{\lambda(x_1^2 + x_2^2)})$ satisfies (P) if $\lambda = (\mu/R)^2$ for some zero μ of $J_1(z)$. Thus, in this case, there are an infinite number of eigenvalues. Conversely, my colleague Carlos Berenstein has shown that if E is a domain for which (P) has infinitely many eigenvalues, then E must be a disc. See [17] for further references.

When μ is a finite measure, the conclusion $\mu = 0$ follows already from the requirement that $\mu(\tau(E)) = 0$ for all translations. Indeed, this last condition is equivalent to the convolution equation $\chi_{-E} * \mu = 0$, which, as we have seen earlier, has $\mu = 0$ as its only solution. Of course, for finite μ there is no longer any need to restrict attention to bounded sets. If E is unbounded, but lies in a half-space, it is still true that the condition $\mu(\sigma(E)) = 0$ for all Euclidean motions implies $\mu = 0$ [13]. On the other hand, in a remarkable paper [9], P.P. Kargaev has constructed a set E of finite measure in R^n and distinct probability measures μ_1, μ_2 such that $\mu_1(\sigma(E)) = \mu_2(\sigma(E))$ for every rigid motion of R^n. Thus, for unbounded sets, the Pompeiu property can fail in the strongest possible sense.

Further results on determining sets for various classes of measures may be found in the interesting survey article [14].

References

1. Carlos A. Berenstein and Lawrence Zalcman, Pompeiu's problem on spaces of constant curvature, J. Analyse Math. 30 (1976), 113-130.

2. Carlos A. Berenstein and Lawrence Zalcman, Pompeiu's problem on symmetric spaces, Comment. Math. Helv. 55 (1980), 593-621.

3. A. S. Besicovitch, A uniqueness theorem and a problem on integration, J. London Math. Soc. 33 (1958), 82-84.

4. Jens Peter Reus Christensen, On some measures analogous to Haar measure, Math. Scand. 26 (1970), 103-106.

5. H. Cramér and H. Wold, Some theorems on distribution functions, J. London Math. Soc. 11 (1936), 290-294.

6. Roy O. Davies, Measures not approximable or not specifiable by means of balls, Mathematika 18 (1971), 157-160.

7. R.J. Gardner and P. McMullen, On Hammer's x-ray problem, J. London Math. Soc. (2) 21 (1980), 171-175.

8. J. Hoffmann-Jorgensen, Measures which agree on balls, Math. Scand. 37 (1975), 319-326.

9. P.P. Kargaev, The Fourier transform of the characteristic function of a set, vanishing on an interval, Math. USSR Sbornik 45 (1983), 397-410.

10. N.A. Sapogov, A uniqueness problem for finite measures in Euclidean spaces, J. Soviet Math. 9 (1978), 1-8.

11. A.A. Shaginyan, On tangential harmonic and gradient approximation, Akad. Nauk Armjan. SSSR Dokl. 77 (1983), 3-5 (Russian).

12. Waclaw Sierpínski, Sur un probleme concernant les ensembles measurables superficiellement, Fund. Math. 1 (1920), 112-115.

13. Alladi Sitaram, A theorem of Ċramer and Wold revisited, Proc. Amer. Math. Soc. 87 (1983), 714-716.

14. Alladi Sitaram, Fourier analysis and determining sets for Radon measures on R^n, Ill. J. Math. 28 (1984), 339-347.

15. Lawrence Zalcman, Analyticity and the Pompeiu problem, Arch. Rat. Mech. Anal. 47 (1972), 237-254.

16. Lawrence Zalcman, Mean values and differential equations, Israel J. Math. 14 (1973), 339-352.

17. Lawrence Zalcman, Offbeat integral geometry, Amer. Math. Monthly 87 (1980), 161-175.

18. Lawrence Zalcman, Uniqueness and nonuniqueness for the Radon transform, Bull. London Math. Soc. 14 (1982), 241-145.