KRZYSZTOF OSTASZEWSKI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA 95616

NONABSOLUTE INTEGRATION IN THE PLANE

In this paper certain results from the dissertation [19] will be presented.

In his topical surveys [29] and [30], Brian Thomson introduced a unified approach to nonabsolute integration on the real line, based on the theory of integral due (in the general setting) to Ralph Henstock ([6], [7], [8]), and (in certain specific settings) to Jaroslav Kurzweil ([13], [14]), and E. J. McShane ([18]).

Following this direction, we consider Henstock integrals in the plane. This requires the notion of a derivation base.

1.1. Definition. Let X be a nonempty set and Ψ a nonvoid class of its subsets. A nonempty class

$$\Delta \subset \mathfrak{P}(X \times \Psi) \tag{1}$$

will be termed a ***derivation base*** on X.

We will usually take X to be $\mathbf{R}^2$ and Ψ — nondegenerate closed intervals, regular intervals, triangles, etc. In [29] and [30] X is taken to the the real line, and Ψ is the class of all closed nondegenerate intervals.

A more general setting is possible. In [1] an integration theory of Henstock type in a locally compact Hausdorff space is presented. A space A equipped with a class $\{I\}$, as in [4] and [31], is also a possibility. Also, [32] presents nonabsolute integration in topological spaces.

A base Δ is called ***trivial*** if $\emptyset \in \Delta$. Unless stated otherwise, all bases considered are nontrivial.

Elements of a base Δ will be denoted by small Greek letters $(\alpha, \beta, \gamma, \ldots)$.

1.2. We will assume that Ψ has the following property: given $I_0, I_1, \ldots, I_n \in \Psi$, and $I_1, \ldots, I_n \subset I_0$,

$$I_0 \setminus (I_1 \cup I_2 \cup \ldots \cup I_n) = J_1 \cup J_2 \cup \ldots \cup J_m \tag{2}$$

1980 *Mathematics Subject Classification.* Primary 26A39

where $J_1, J_2, \ldots, J_m$ are nonoverlapping elements of Ψ (since we work in $\mathbb{R}^2$ the meaning of "nonoverlapping" will be clear).

1.3. Definition. We say that a finite class D of elements of Ψ is a *division* if its elements are nonoverlapping.

A *partition* is a class $\pi \in \mathfrak{P}(X \times \Psi)$ such that

$$D = \{I \in \Psi : (x, I) \in \pi\} \tag{3}$$

has exactly as many elements as π and is a division.

D is a division of an element I_0 of Ψ if $\bigcup_{I \in D} I = I_0$. Similarly, π is a partition of I_0 if $\bigcup_{(x,I)\in\pi} I = I_0$.

If $F : X \times \Psi \to \mathbb{R}$ and π is a partition then we will write

$$F(\pi) = \sum_{(x,I)\in\pi} F(x, I). \tag{4}$$

1.4. Definition. A base Δ is *filtering down* if for every $\alpha_1, \alpha_2 \in \Delta$ there exists an $\alpha \in \Delta$ such that $\alpha \subset \alpha_1 \cap \alpha_2$.

Δ has the *partitioning property* if for every $I \in \Psi$ and every $\alpha \in \Delta$ there exists a partition $\pi \subset \alpha$ of I.

1.5. Definition. A base Δ is *finer* than a base Δ' if for every $\alpha' \in \Delta'$ there exists an $\alpha \in \Delta$ such that $\alpha \subset \alpha'$.

If Δ is finer than Δ' then we write $\Delta \preceq \Delta'$.

If $\Delta \preceq \Delta'$ and $\Delta \succeq \Delta'$ then we will say that Δ and Δ' are *equivalent* and write $\Delta \simeq \Delta'$.

1.6. Definition. Δ has a *local character* if for every $\{\beta_x\} \in \prod_{x\in X} \Delta[\{x\}]$ there exists an $\alpha \in \Delta$ such that for every $x \in X$

$$\alpha[\{x\}] \subset \beta_x. \tag{5}$$

1.7. Definition. Let $I_0 \in \Psi$, $F : I_0 \times \Psi \to \mathbb{R}$. We define the *Henstock integral* of F with respect to Δ over I_0 as a number $(\Delta) \int_{I_0} F$ such that for every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ such that for every partition $\pi \subset \alpha$ of I_0

$$\left| F(\pi) - (\Delta) \int_{I_0} F \right| \le \varepsilon. \tag{6}$$

1.8. Definition. Let Δ^1 be a derivation base on X and Δ^2 — a base on Y. Assume that Δ^1 and Δ^2 have local character. Let $\Psi^1 \subset \mathfrak{P}(X)$ and $\Psi^2 \subset \mathfrak{P}(Y)$ be the corresponding classes of "intervals".

Set

$$\Psi = \{I \times J : I \in \Psi^1, J \in \Psi^2\} \quad \text{and} \quad Z = X \times Y. \tag{7}$$

$\Delta \subset \mathfrak{P}(Z \times \Psi)$ will be termed the *product base* of Δ^1 and Δ^2 (written as $\Delta = \Delta^1 \times \Delta^2$) if for every $\alpha \in \Delta$ there exist functions

$$\begin{aligned} X \ni x \mapsto \alpha_x^2 \in \Delta^2, \\ Y \ni y \mapsto \alpha_y^1 \in \Delta^1 \end{aligned} \tag{8}$$

such that $(z, P) \in \alpha$ if and only if

$$z = (x, y) \quad \text{and} \quad P = I \times J \tag{9}$$

where

$$(x, I) \in \alpha_y^1 \quad \text{and} \quad (y, J) \in \alpha_x^2. \tag{10}$$

2.1. Definition. Let Φ stand for the class of all nondegenerate closed intervals in $\mathbf{R}^2$. Take $X = \mathbf{R}^2$ and $\Psi = \Phi$.

Let P be the class of all real-valued, positive functions on $\mathbb{R}^2$.

The *Kurzweil base* Δ_1 consists of all α_p, where $p \in P$ and

$$\alpha_p = \Big\{(x, I) \in \mathbf{R}^2 \times \Phi : x \in I,\ I \subset D\big(x, p(x)\big)\Big\}. \tag{11}$$

If we drop the condition "$x \in I$" in (11), we get Δ_1^*, which will be called the *weak Kurzweil base.*

If we replace "$x \in I$" in (11) by "x is a vertex of I", then we get $\widetilde{\Delta}_1$ which will be called the *modified Kurzweil base.*

2.2. Definition. Let $I \in \Phi$. We define its *norm* $n(I)$ as the length of its longer side.

If $\{I_1, \ldots, I_n\}$ is a finite subclass of Φ then the *norm* $n\big(\{I_1, \ldots, I_n\}\big)$ is defined to be the greatests of all $n(I)$ for $I \in \{I_1, \ldots, I_n\}$.

2.3. Definition. Let $I \in \Phi$. We define its *regularity* as the number

$$r(I) = \frac{\lambda(I)}{\big(n(I)\big)^2} \tag{12}$$

(see [28], p. 106), where $\lambda(I)$ is the area of I.

It is easy to see that $0 < r(I) \leq 1$.

2.4. Definition. Let $\varrho \in (0,1)$. We will say that I is *ϱ-regular* if $r(I) \geq \varrho$. And we will write Φ_ρ for the class of all elements of Φ which are ϱ-regular.

2.5. Definition. Now let $\Psi = \Phi_\rho$, and $X = \mathbf{R}^2$.

For $p \in \mathcal{P}$ let

$$\alpha_p^\varrho = \Big\{(x, I) \in \mathbf{R}^2 \times \Phi_\rho : x \in I,\ I \subset D\big(x, p(x)\big)\Big\}, \tag{13}$$

and

$$\Delta_2^\varrho = \{\alpha_p^\varrho : p \in \mathcal{P}\}. \tag{14}$$

We will call Δ_2^ϱ the *Kempisty ϱ-base*. If we drop the assumption "$x \in I$" in (13), we get the *weak Kempisty ϱ-base* $\Delta_2^{\varrho\star}$.

We will usually fix a $\varrho \in (0,1)$ and write Δ_2 and $\Delta_2^\star$ instead of Δ_2^ϱ and $\Delta_2^{\varrho\star}$.

2.6. Definition. If we replace intervals in the definition 2.1 by triangles (compare this with the work in [20], [21], [23], and [26]), the base so obtained will be called the *Pfeffer base* (*weak Pfeffer base*) and denoted by Δ_3 ($\Delta_3^\star$).

2.7. Definition. We will say that an interval I is *generated by x_1 and x_2*, elements of $\mathbf{R}^2$, if x_1 and x_2 are opposite vertices of I.

2.8. Definition. Let a system

$$\mathcal{N} = \big\{N(x) : x \in X\big\} \tag{15}$$

of nontrivial filters $N(x)$ of subsets of X, converging to $x \in X$, be given. A *filtered base* Δ generated by it is defined as

$$\Delta = \Big\{\alpha_\eta : \eta \in \prod_{x \in X} N(x)\Big\} \tag{16}$$

where

$$\alpha_\eta = \big\{(x, I) : I \text{ is generated by } x \text{ and some } x' \in \eta(x)\big\}. \tag{17}$$

An element η of the Cartesian product in (16) will be called a *choice.*

2.9. Observation. Let $\mathcal{T}$ be a Hausdorff topology on the plane. Then

$$\mathcal{N}(x) = \{G \in \mathcal{T} : x \in G\} \tag{18}$$

is a filter satisfying the assumptions of 2.8. Therefore any Hausdorff topology naturally generates a filtered base.

2.10. Definition. Consider the three density topologies in $\mathbb{R}^2$ defined in [5], i.e., the ordinary, strong, and "tilde" topologies. The filtered bases generated by them with be denoted by Δ_7, Δ_6, and Δ_5, respectively.

2.11. Definition. If $\mathcal{D}$ is the density topology on the line (see [5]), then we can define, just as it was done in 2.9, the filtered base on $\mathbb{R}$ generated by it. This base (see [29], p. 85) will be denoted by $\mathcal{A}$ and called the *approximate base* on the real line.

$\mathcal{A}$ is filtering down, has local character and the partitioning property.

2.12. Definition.

$$\Delta_4 = \mathcal{A} \times \mathcal{A}. \tag{19}$$

3.1. In [19] we investigate Henstock integrals generated by the bases listed here. We compare them with integrals of Lebesgue, Perron (see [15]), Kempisty (see [9], [10], [11], and [12]), Mawhin (see [16] and [17]), Pfeffer (see [23], [25], and [27]), and Chelidze-Dzhvarsheishvili (see [2] and [3]).

The relationships among them found are presented graphically in a diagram on the next page.

In the diagram, integration theories are represented by the bases generating them, or by the names of their inventors. Arrows point to the more general theories. +(condition) means that the condition stated is necessary for the relationship. *CSS* denotes continuity in the sense of Saks, defined in [19] (definition 2.8.4), *SMC* — special assumption on majorants and minorants of theorem 4.5.4 of [19], and *SDC* — special decomposition condition of theorem 5.5.5 of [19].

$$
\begin{tikzcd}
\Delta_3^* \arrow[r, shift left] & \text{Lebesgue} \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift right] & \Delta_2^* \arrow[l, shift left] \\
 & \Delta_1^* \arrow[u, shift right] \arrow[d] & \\
\text{Perron} \arrow[r, shift left] \arrow[d, shift right] & \Delta_1 \arrow[l, shift left] \arrow[r] \arrow[d] & \Delta_4 \arrow[d] \\
\tilde{\Delta}_1 \arrow[u, shift right] \arrow[d] & \text{Pfeffer} \arrow[d] & \Delta_5 \arrow[d] \\
\text{Mawhin} \arrow[r] & \Delta_2 \arrow[d, shift left, "\begin{smallmatrix}+CSS\\+SMC\end{smallmatrix}"] & \Delta_6 \arrow[d] \\
 & \text{Kempisty} \arrow[u, shift left] & \Delta_7 \\
 & & \text{Chelidze-Dzhvarsheĭshvili} \arrow[u, "+SDC"']
\end{tikzcd}
$$

REFERENCES

[1] S. I. Ahmed, and W. F. Pfeffer, A Riemann integral in a locally compact Hausdorff space, a preprint.

[2] W. G. Chelidze, Double Denjoy integrals, *Trudy Mat. Inst. Akad. Nauk GSSR* **15** (1947), 155–242.

[3] W. G. Chelidze, and A. G. Dzhvarsheĭshvili, *Theory of the Denjoy integral and some of its applications*, Tbilisi University, 1978.

[4] L. Cesari, Quasi-additive set functions and the concept of integral over a variety, *Trans. Amer. Math. Soc.* **102** (1962), 94–113.

[5] C. Goffman, C. J. Neugebauer, and T. Nishiura, Density topology and approximate continuity, *Duke Math. J.* **28**(4) (1961), 497–506.

[6] R. Henstock, Integration, variation, and differentiation in division spaces, *Proc. Royal Irish Acad. Sec.* **78A** (1978), 69–85.

[7] R. Henstock, *Linear Analysis*, Butterworths, 1978.

[8] R. Henstock, *Theory of integration*, Butterwoths, 1963.

[9] S. Kempisty, *Fonctions d'intervalle non additive*, Actualités Sc. et Ind., No. 824, Hermann et Cie, Paris, 1939.

[10] S. Kempisty, Intégrale Denjoy-Stieltjes d'une fonction de deux variables, *C. R. Acad. Sc. Paris* **202** (1936), 1241–1244.

[11] S. Kempisty, Sur la totalisation des fonctions de deux variables, *C. R. Acad. Sc. Paris* **198** (1934), 2060–2065.

[12] S. Kempisty, Sur les fonctions absolument continues d'intervalle, *Fund. Math.* **27** (1936), 10–37.

[13] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czech. Math. J.* **7**(82) (1957), 418–449.

[14] J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner-Texte zur Mathematik, No. 26, Leipzig, 1980.

[15] J. Mařík, Základy theorie integrálu v euklidových prostorech, I-III, *Časopis Pešt. Mat.* **77** (1952), 1–51, 125–145, 267–300.

[16] J. Mawhin, Generalized multiple Perron integrals and the Green-Goursat theorem for differetiable vector fields, *Czech. Math. J.* **31**(106) (1981), 614–632.

[17] J. Mawhin, Generalized Riemann integrals and the divergence theorem for differentiable vector fields, pp. 704–714 in *Proceedings of the International Christoffel Symposium*, Birkhäuser, Basel, 1981.

[18] E. J. McShane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Memoir Amer. Math. Soc., No. 88, 1969.

[19] K. Ostaszewski, *Henstock integration on the plane*, Ph. D. dissertation, University of Washington, 1985.

[20] W. F. Pfeffer, An integral in topological spaces I, *J. Math. Mech.* **18** (1969), 953–972.

[21] W. F. Pfeffer, An integral in topological spaces II, *Math. Scand.* **27** (1970), 77–104.

[22] W. F. Pfeffer, A Riemann integral and the divergence theorem, a preprint.

[23] W. F. Pfeffer, On a definition of the integral in topological spaces, I-II, *Časopis Pešt. Mat.* **89** (1964), 129–147, 257–277.

[24] W. F. Pfeffer, *The Divergence Theorem*, Technical Report No. 64, Department of Mathematical Sciences, University of Petroleum and Minerals, Dhahran, Saudi Arabia, April 1984.

[25] W. F. Pfeffer, The divergence theorem, a preprint.

[26] W. F. Pfeffer, The existence of locally fine simplicial subdivisions, *J. Australian Math. Soc. Sec. A* **33** (1982), 114–124.

[27] W. F. Pfeffer, Une intégrale Riemannienne et le théorème de divergence, *C. R. Acad. Sc. Paris* **299** (1984), 299–301.

[28] S. Saks, *Theory of the Integral*, sec. ed., New York, 1937.

[29] B. S. Thomson, Derivation bases on the real line, *Real Anal. Exchange* **8**(1) (1982–83), 67–207.

[30] B. S. Thomson, Differentiation bases on the real line, *Real Anal. Exchange* **8**(2) (1982–83), 278–442.

[31] G. Warner, The Burkill-Cesari integral, *Duke Math. J.* **35** (1968), 61–78.

[32] W. J. Wilbur, *On nonabsolute integration in topological spaces*, Ph. D. dissertation, University of California at Davis, 1967.

Professor Foran's report appears in the Research Articles section of this issue of the <u>Real Analysis Exchange</u>