J. Marshall Ash[1], Mathematical Sciences Department, DePaul University, Chicago, IL 60614, USA

Very Generalized Riemann Derivatives, Generalized Riemann Derivatives and Associated Summability Methods

I. VERY GENERALIZED RIEMANN DERIVATIVES

0. Generalized Riemann derivatives.

Let f be a real valued function of a real variable. The nth Riemann derivative of f is

$$R_n f(x) := \lim_{h\to 0} \frac{\sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} f(x+(-\frac{n}{2}+i)h).}{h^n}$$

The first two special cases

$$R_1 f(x) = \lim_{h\to 0} \frac{-f(x-\frac{h}{2}) + f(x+\frac{h}{2})}{h}$$

and

$$R_2 f(x) = \lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

are the well known symmetric and Schwarz derivatives.

The generalized Riemann derivative which was the subject of my 1966 thesis[1] is

(1) $$D_n(\mathbf{b},\mathbf{a})f(x) := \lim_{h\to 0} \frac{\Delta_n(h;\mathbf{b},\mathbf{a})f(x)}{h^n}$$

where

(2) $$\Delta_n(h;\mathbf{b},\mathbf{a})f(x) := \sum_{i=0}^{n+e} a_i f(x+b_i h)$$

[1]The research presented here was supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences, DePaul University

where e is a non-negative integer which I will call the excess and the a_i's and b_i's are real numbers. Here we insist upon the n+1 consistency conditions

$$\Sigma\, a_i b_i^j = \begin{Bmatrix} 0 & j = 0,1,\ldots,n-1 \\ n! & j = n \end{Bmatrix}. \tag{3}$$

For notational convenience I will always assume $b_o < b_1 < \ldots < b_{n+e}$.

1. Relations between different generalized derivatives.

To see why these conditions are imposed let $f^{(n)}(x_o)$ exist so that

$$f(x_o + k) = \sum_{j=0}^{n} \frac{f^{(j)}(x_o)}{j!} k^j + o(k^{n+1}).$$ (Here and throughout $g(h)=o(h^\alpha)$ means $\frac{g(h)}{h^\alpha} \to 0$ as $h \to 0$.) This expansion is a slightly souped up version of Taylor's theorem which is due to de la Vallee-Poussin. Professor A. Zygmund showed it to me. Substitute this into (1) with k equal successively $b_o h, b_1 h, \ldots, b_{n+e} h$ to get

$$\begin{aligned} \Sigma\, a_i f(x_o+b_i h) &= \sum_i a_i [\sum_j f^{(j)}(x_o)(b_i h)^j] + o(h^n) \\ &= \Sigma \frac{f^{(j)}(x_o)}{j!} h^j\, [\Sigma a_i b_i^j] + o(h^n) \\ &= \frac{f^{(n)}(x_o)}{n!} [n!] h^n + o(h^n). \end{aligned} \tag{4}$$

Divide by h^n and let $h \to 0$. We get $D_n f(x_o)$ so that our derivatives are extensions of the usual ones. Very simple examples show these extensions to be strict. For example, $a(x) = |x|$ has $R_1 a(0) = 0$ while $a'(0)$ does not exist, and $s(x) = \text{signum}(x)$ has $R_2 s(0) = 0$ while $s'(0)$ and $s''(0)$ do not exist.

The reason for calling e the excess is that if e=0 then the b_i's determine the a_i's via condition (2). Explicitly,

$$a_i = \frac{n!}{\prod_{j\neq i}(b_i - b_j)}. \tag{5}$$

To see this, let $L_i(x) := \frac{\prod_{j\neq i}(x - b_j)}{\prod_{j\neq i}(b_i - b_j)}$ be the Lagrange interpolating polynomial so that $L_i(b_i) = 1$ and $L_i(b_j) = 0$ when $j \neq i$. Then from (2) it is immediate that $\Delta_n(1;\mathbf{b},\mathbf{a})L_i(0) = a_i$. On the other hand, $L_i(x) = [\Pi(b_i-b_j)]^{-1}x^n$ + lower powers of x, whence the nth ordinary derivative of L_i is the constant $n![\Pi(b_i-b_j)]^{-1}$. The Taylor expansion out to h^n is exact, i.e., without higher order terms, for polynomials of degree n, so that equations (4) show that for all x and h, $\frac{\Delta_n(h;\mathbf{b},\mathbf{a})L_i(x)}{h^n}$ is equal to this constant. Setting $x = 0$ and $h = 1$ proves (5). In particular, you can't make a first derivative without at least 2 terms, nor a second without at least 3, nor an n-th without at least n+1 points.

On the other hand even if all b_i's are fixed, if $e > 0$ you can choose e of the a_i's freely; then conditions (2) determine the rest.

Denjoy looked at the case of excess = 0.[11] I seem to have been the first to look at $e > 0$ systematically although particular cases have shown up in numerical analysis before.

The n-th Peano derivative f_n is a generalization of the ordinary derivative lying midway between the ordinary n-th derivative and $D_n f(x)$. By definition $f_n(x_o)$ exists if n other numbers $f_o(x_o), f_1(x_o), \dots, f_{n-1}(x_o)$ also exist so that

$$f(x_o+h) = f_o(x_o) + f_1(x_o)h + \dots + f_n(x_o)\frac{h^n}{n!} + o(h^n).$$

Note that f is continuous at x if $f_o(x) = f(x)$ and f is differentiable at x if and only if $f_1(x)$ exists. Then $f'(x) = f_1(x)$. The classic example showing f_2 to be a strict extension of f" is $x^3 \sin\frac{1}{x}$ at the point x=0. Note that what we proved above shows each D_n to be an extension of f_n. Also note that the examples a(x) and s(x) show R_1 a strict extension of f_1($=f'$) and R_2 a strict extension of f_2. Again every D_n (except D_1 with $a_o=0$, $a_1=1$) is a strict extension of the corresponding f_n.

However the implication $\exists f_n \rightarrow \exists D_n$ _is_ reversible provided we are willing to throw away a set of Lebesgue measure 0. This was the main result of my 1966 PhD thesis.[1]

If $n \geq 2$, one cannot return from f_n to $f^{(n)}$ even on an almost everywhere basis. This question was discussed by Oliver in 1953. [15] He does prove that $\exists f_n \rightarrow \exists f^{(n)}$ provided $f_n(x)$ is a bounded function on an interval as well as several other interesting results.

There is also a derivative, designated d_2 in [2], which lies between f_2 and every D_2 in an almost everywhere sense.

Most of these notions and results go through in an L^p metric sense.[1],[2] Another way to return from D_n to f_n does work at a single point. This time assume that f is measurable and that <u>every</u> $D_n f(x_o)$ exists. Then it does follow that $f_n(x_o)$ exists. To improve on this result one should cut down on the number of Riemann derivatives assumed existent at x_o. Coupling the results of a 1969 paper - A Characterization of the Peano derivative - and a 1974 paper with Erdos and Rubel we have the following result.[2],[5]

Let $\Delta_1(h) := f(x+h)-f(x)$,

$\Delta_2(a_1,h) := \Delta_1(a_1h) - a_1\Delta_1(h) = f(x+a_1h) - a_1f(x+h) + (a_1-1)f(x), \ldots,$

$\Delta_n(a_1,\ldots,a_{n-1};h) := \Delta_{n-1}(a_1,\ldots,a_{n-2};a_{n-1}h) - a_{n-1}^{n-1}\Delta_{n-1}(a_1,\ldots,a_{n-2};h)$

and let $D_n(\mathbf{a})(x) := \lim_{h\to 0} \dfrac{\Delta_n(\mathbf{a};h)}{h^n}$ (The a_i's are not 0, 1 or -1.) If f is measurable, and if whenever $\mathbf{a} \in M^{n-1}$, $D_n(\mathbf{a})$ exists at $x = x_o$, and if M is "thick" enough; then $f_n(x_o)$ exists. The thickness of the set M determines how good this theorem is. Easy examples show that it is not enough for M to be countably infinite, nor for M to consist solely of positive numbers. If M has positive measure and contains a negative number then M is thick enough.

At x=0 the second derivative R_2 differentiates s(x) but not a(x), while the second derivative

$P_2f(x) := \lim_{h\to 0} \dfrac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$ does not differentiate s(x), but does differentiate a(x) since looking only forward a(x) is a straight line and looking only backwards a(x) is also a straight line. However Patrick J. O'Connor, in an unpublished 1969 PhD thesis at Connecticut Wesleyan shows that whenever two generalized Riemann n-th derivatives both exist at a point, they must agree.[14]

The idea of his proof is quite nice. If $D_n = \lim_{h\to 0} \Sigma\, a_i f(x+b_i h)$ and $D'_n = \lim_{h\to 0} \Sigma a'_j f(x+b'_j h)$, form $D_n \circledast D'_n := \lim_{h\to 0} \frac{1}{n!} \sum_{i,j} a_i a'_j\, f(x+b_i b'_j h)$. It is then easy to prove that $D_n \circledast D'_n$ is also a generalized Riemann derivative and that it agrees with both D_n and D'_n.

2. <u>Numerical</u> <u>Analysis</u>.

Generalized Riemann derivatives have had application in numerical analysis. The symmetric derivative R_1 is "better" for approximation purposes than the ordinary derivative in the sense that for fixed h and very smooth f,

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{1}{2} f''(\xi)h \text{ while}$$

$$\frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} = f'(x) + \frac{1}{48} f^{(3)}(\xi)h^2 \text{ and the error}$$

term $\frac{1}{48} f^{(3)}(\xi)h^2$ is "sort of smaller" than $\frac{1}{2} f''(\xi)h$. Notice that to make the comparison fair I normalize and keep $b_2 - b_1 = 1$ in both cases. So to compare approximations to the first derivative based on 2+e function evaluations I fix h and look at differences

$h^{-1} \sum_{i=0}^{e+1} a_i f(x+b_i h) = \Delta(\mathbf{b},\mathbf{a})f(x)$ subject to this normalization

$b_{i+1} - b_i \geq 1$ for all $i \geq 0$. If 2 such differences give for good f

$$\Delta(\mathbf{b},\mathbf{a})f(x) = f'(x) + c_r f^{(r)}(x)h^{r-1} + O(h^r)$$

and $$\Delta(\mathbf{b}',\mathbf{a}')f(x) = f'(x) + c_s f^{(s)}(x)h^{s-1} + O(h^s)$$

<u>define</u> $\Delta(\mathbf{b},\mathbf{a})$ to be better than $\Delta(\mathbf{b}',\mathbf{a}')$ if either $r > s$, or $r=s$ and $c_r < c_s$.

Then indeed $\mathbf{b} = (-\frac{1}{2}, \frac{1}{2})$ gives the best 2 point difference. Again the best 4 point difference has $\mathbf{b} = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ which is still no surprise. Again the answer you would guess for 6, 8, or any even number of points is correct. However, for 3 points the best $\mathbf{b}$ is

$$\mathbf{b} = (\frac{1}{\sqrt{3}} -1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} +1) \approx (-.423, .577, 1.577),$$
$$= (\alpha_3 -1, \alpha_3, \alpha_3 +1),$$

for 5 points

$$\mathbf{b} = (\alpha_5 -2, \alpha_5 -1, \alpha_5, \alpha_5 +1, \alpha_5 +2)$$

where $\alpha_5 = \sqrt{15-\sqrt{145}/10} \approx .544$, and for 2k+1 points

$\mathbf{b} = (\alpha_{2k+1} - k, \ldots\ldots, \alpha_{2k+1}, \ldots\ldots, \alpha_{2k+1} + k)$ where the α_n satisfy $\frac{1}{2} < \alpha_n < \frac{1}{2} + \frac{1}{4n}$, $n=3,5,\ldots.$ and α_n is determined as the smallest positive zero of $\frac{d}{dx}(\prod_{i=-k}^{k} (x-i)) = 0$. The choice of $\mathbf{b}$ and the approximating conditions

$$\Sigma\, a_i = 0$$
$$\Sigma\, a_i b_i = 1$$
$$\Sigma\, a_i b_i^j = 0 \qquad j = 2,3,\ldots,n-2$$

determine $\mathbf{a}$ by linear algebra. This choice is unique up to the trivial inversion $(\mathbf{b},\mathbf{a}) \rightarrow (-\mathbf{b},-\mathbf{a})$.

A similar situation occurs for the second derivative. Here the starting point is that R_2 gives the best 3 point difference. The results are similar to those above. Now the best 3,5,7,... point differences are based on the obvious symmetric choices of b while the even **b**'s are more interesting with the best 4 point b being

$$\mathbf{b} = (\beta_4-2,\ \beta_4-1,\ \beta_4,\ \beta_4+1),\ \beta_4 = (1 + \sqrt{5/3}\,)/2 \approx 1.145$$ and so on. In a 1981 Math. Comp. paper Roger Jones and I work out the 3 point first derivative case which remains optimal even when roundoff error is taken into account [7]. The general results I just mentioned are detailed in a 1984 paper in Estratto de Calcolo with Svante Janson and Roger Jones.[9]

<u>Question 1</u>. Extend these results to n > 2. (Even n=3 was too hard for us.)

3. <u>Classification Questions</u>

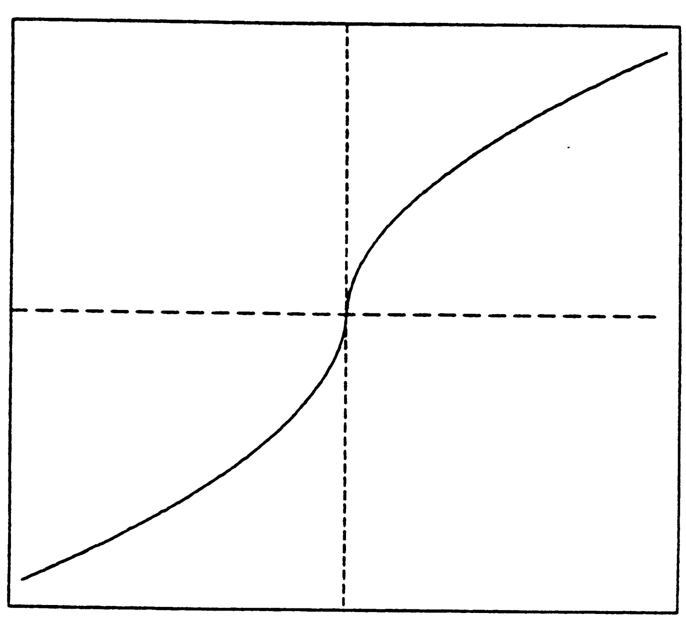
A very interesting example is provided by the first derivative

$$O_1 f(x) := \lim_{h\to 0} \frac{7f(x+3h) - 13f(x+4h) + 6f(x+\frac{16}{3}h)}{h}$$

and the function

$$f(x) := \operatorname{sgn}(x)|x|^{\log_{4/3}(7/6)} - x.$$

This example is given by Patrick O'Connor in his thesis.[14] Since $p := \log_{4/3}(7/6) = \frac{\ln(7/6)}{\ln(4/3)} \approx .54$, $\operatorname{sgn}(x)|x|^p$ looks like $\operatorname{sgn}(x)\sqrt{|x|}$,



and f looks about the same. But then $O_1 f(x) = f'(x)$ whenever $x \neq 0$ and direct calculation shows that $O_1 f(0) = -1$. This example has a lot of shock value for me. Here is the graph of O_1

We have a non-Darboux derivative. We also have an everywhere increasing, everywhere differentiable (with respect to O_1) function whose derivative is negative at a point.

On the other hand consider the symmetric derivative R_1. This derivative's existence does force a function to be Darboux. If a strictly increasing function has an everywhere existing symmetric derivative, then that derivative is positive. These two properties also hold for f'. We thus have at least 2 classification problems.

Question 2. Which generalized Riemann derivatives are Darboux? That is, for which D_1 does the existence of $D_1 F(x) =: f(x)$ at every point x force f to have the intermediate value property?

Question 3. For which D_1 does f increasing on $(a-\epsilon, a+\epsilon)$ and $D_1 f(a)$ existing force $D_1 f(a) > 0$?

Notice that for both questions O_1 is in the bad class, while R_1 and $\frac{d}{dx}$ are both in the good class.

4. Further generalization.

Let us now justify the "very" in the title of the talk. By the very generalized Riemann derivative $D_n^+(\mathbf{b},\mathbf{a})$ I mean the same thing as before except that the limit is now one sided, so

$$D_n(\mathbf{b},\mathbf{a}) f(x) = \lim_{h \to 0^+} \frac{\Delta_n(h;\mathbf{b},\mathbf{a}) f(x)}{h^n}.$$

There is no need for a D_n^- to be defined since for example one has

$$\frac{\Sigma a_i f(x+b_i h)}{h} = \lim_{h\to 0^-} \frac{\Sigma(-a_i)f(x+(-b_i)(-h))}{(-h)}$$

$$= \lim_{h\to 0^+} \frac{\Sigma(-a_i)f(x+(-b_i)h)}{h} = D_1^+(-\mathbf{b},-\mathbf{a}).$$

One could go on to define objects similar to Dini numbers such as

$$\limsup_{h\to 0^+} \frac{\Delta_n(h;\mathbf{b},\mathbf{a})f(x)}{h^n}$$

but I have not done anything in this direction.

It is obvious that D_n^+ is an extension of D_n, i.e. that if $D_n(\mathbf{b},\mathbf{a})f(x_o)$ exists so does $D_n^+(\mathbf{b},\mathbf{a})f(x_o)$ and the two are then equal. The extension is usually proper. Note that $R_n^+ = R_n$ and more generally enough symmetry in $\mathbf{a}$ and $\mathbf{b}$ will make $D_n^+ = D_n$. Probably one could prove that $\{(b_i,a_i)\} = \{(-b_i,-a_i)\}$ for n odd and $\{(b_i,a_i)\} = \{(-b_i,a_i)\}$ for n even is a necessary and sufficient condition for the extension to be improper, i. e., for $D_n^+ = D_n$ to hold.

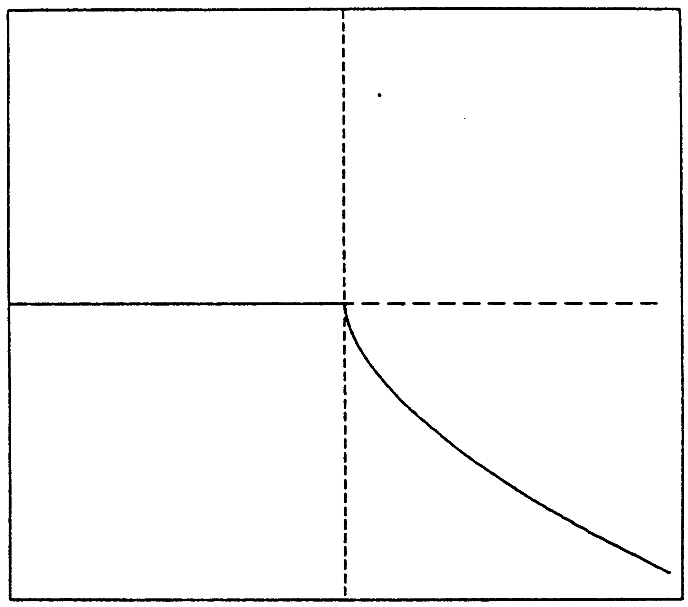
The function $a(x) = |x|$ has $(\frac{d}{dx})^+a(0)=1$ although $(\frac{d}{dx})a(0)$ doesn't exist. A more interesting example is the second derivative $A_2^+f(x) := \lim_{h\to 0^+} \frac{(2/3)f(x+2h) - f(x+h) + (1/3)f(x-h)}{h^2}$. Note that $\frac{2}{3} - 1 + \frac{1}{3} = 0$, $\frac{2}{3}(2) - 1(1) + \frac{1}{3}(-1) = 0$ and $\frac{2}{3}(2)^2 - 1(1)^2 + \frac{1}{3}(-1)^2 = 2$. Then consider the function $u(x) = \begin{cases} 0 & x < 0 \\ -x^{\log_2(3/2)} & x \geq 0 \end{cases}$. For $h > 0$,

$$\frac{(2/3)u(0+2h)-u(0+h)+(1/3)u(0-h)}{h^2} = \frac{-[(2/3)(2h)^q - (h)^q]}{h^2} =$$

$$\frac{-[(2/3)\cdot 2^{\log_2(3/2)} - 1]}{h^{2-q}} = 0 \text{ , so that } A_2^+u(0) = 0.$$

Clearly for $x \neq 0$, $A_2^+ u(x) = u''(x) = \begin{Bmatrix} 0 & x < 0 \\ q(1-q)x^{q-2} & x > 0 \end{Bmatrix}$. A similar calculation for $h < 0$ shows that $A_2 u(0)$ does not exist.

Again $q := \log_2(\frac{3}{2}) = \frac{\ln(3/2)}{\ln 2} \approx .58$ so $x^{\log_2(3/2)}$ looks like $\sqrt{x}$ for positive x. Here is u.



If one allows $h \to 0^-$ as well, then the situation of continuous non-convex f with $A_2 f \geq 0$ everywhere does not arise. One reason to study A_2^+ is the following. The 0 excess very generalized second Riemann derivatives may be classified as

type I if $b_0 < b_1 = 0 < b_2$;

type II if $b_0 < 0 < b_1 < b_2$ or if $b_0 < b_1 < 0 < b_2$; and

type III if $b_0 < b_1 < b_2 \leq 0$ or if $0 \leq b_0 < b_1 < b_2$.

I think that all the questions I will raise in studying A_2^+ will have easy answers for type I and type III derivative and that A_2^+ will prove to be a prototype for all those of type II. We will see more of u and A_2^+ shortly.

II. GENERALIZED RIEMANN DERIVATIVES AND ASSOCIATED SUMMABILITY METHODS

5. Generalized differentiation and uniqueness for trigonometric series.

Let $T = \Sigma\, c_n e^{inx}$ be a trigonometric series. Suppose that at every $x \in [0,2\pi)$ $T(x) := \lim_{N\to\infty} \Sigma_{-N}^{N}\, c_n e^{inx} = 0$. Then all $c_n=0$. This is the fundamental theorem in the subject. It was announced by Riemann in 1854 and the last detail of his proof was supplied in a letter from H.A. Schwarz to Cantor who published it in 1870.[10],[16],[17]

Theorem R. If F is continuous and $R_2F=0$ everywhere, then F is a line.

This theorem is immediate from a lemma.

Lemma R. If F is continuous and $R_2F \geq 0$ everywhere then F is convex.

Consider the following statement.

"Lemma" A. If F is continuous and $A_2^+F \geq 0$ everywhere, then F is convex.

As the continuous non-convex u enjoys $A_2^+\, u \geq 0$ for all x, this statement is false.

However, we are left with the following open question.

"Theorem" A. If F is continuous and $A_2^+F = 0$ everywhere, then F is linear.

Question 4. Is "Theorem" A true?

This question is very hard. Why does it matter? On the one hand, theorem R is the cornerstone of the entire theory of uniqueness. There are many open questions concerning multiple trigonometric series whose resolution would be easy if higher dimensional analogues of Theorem R were available. For example suppose $T(x,y,z)$ converges unrestrictedly rectangularly to 0, that is, suppose

$$\lim_{L,M,N\to\infty} \sum_{l=-L}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} c_{lmn} e^{i(lx+my+nz)} = 0, \text{ at every } (x,y,z).$$

No one knows if it then follows that all c_{lmn} are 0. On the other hand, Theorem R has only one known proof, namely via Lemma R. To extend Theorem R to higher dimensional settings it could be useful to have another proof. A proof of "Theorem" A couldn't use the false "Lemma" A and so would probably also yield a genuinely new proof of Theorem R.

Another question related to uniqueness is

<u>Question 5</u>. Let $F(x,y)$ be continuous and suppose

$$0 = \lim_{h,k\to 0} \left\{ \begin{array}{l} F(x-h,y+k) - 2F(x,y+k) + F(x+h,y+k) \\ -2F(x-h,y) + 4F(x,y) - 2F(x+h,y) \\ +F(x-h,y-k) - 2F(x,y-k) + F(x+h,y-k) \end{array} \right\} \cdot \frac{1}{h^2k^2}$$

at each (x,y). Is F then necessarily of the form $F(x,y) = (ax + b) + (cy + d)$ where a and b are functions of only y, and c and d are functions of only x? See my paper with Welland or my survey article in my book for some details and partial results about this.[3],[6]

A related question is

<u>Question 6</u>. It follows easily from Theorem R that if $\frac{1}{h} \int_0^h |f(x+t)-f(x-t)|dt = o(h)$ at all points x, then f is constant. Prove this without invoking Lemma R.

This would follow if a function with everywhere 0 symmetric approximate derivative could be shown to be constant. A positive resolution of question 6 will necessarily also provide a new proof of Riemann's uniqueness theorem.[4]

6. Generalized Differentiation and Summability.

In an attempt to prove "Theorem" A I was led to a related summability result. Let $F(x) = \Sigma\, c_n e^{inx}$ be a continuous function. Form the distributional second derivatives $F'' := \Sigma\, (in)^2 c_n e^{inx}$. An elementary computation shows

$$\frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = \Sigma\, (in)^2 c_n e^{inx} \left(\frac{\sin nh}{nh}\right)^2 .$$

By definition $R_2F(x) := \lim_{h\to 0}$ (L.H.S.) and by definition the series F'' is summable (R,2) to s if $s = \lim_{h\to 0}$ (R.H.S.). Thus theorem R can be restated by saying that a continuous function whose distributional second derivative is summable (R,2) everywhere to 0 is linear. Similarly the derivative A_2^+ corresponds to a method of summability, call it summability A_2^+. There is a theorem of Kuttner [13] that summability (R,2) implies Abel summability and a theorem of Verblunsky [17] stating that if $\Sigma c_n e^{inx}$ is Abel summable to 0 everywhere and $c_n=o(n)$ then all $c_n=0$. I hoped to show "Theorem" A by first showing summability A_2^+ implies Abel summability, then controlling the coefficients, and finally applying Verblunsky's theorem.

So define a series Σa_n to be summable A_2^+ to s if

$$\lim_{h\to 0^+} \sum_{n=-\infty}^{\infty} a_n \gamma(inh) = s \quad \text{where}$$

$$\gamma(t) = \frac{(2/3)e^{2t} - e^{t} + (1/3)e^{-t}}{t^2}.$$

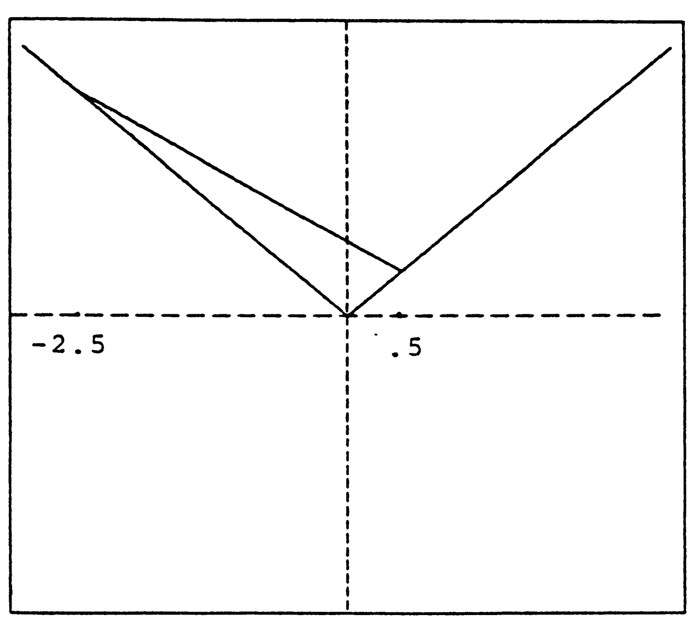
As with the Riemann situation we have $A_2^+F(x)$ exists if and only if the twice formally differentiated Fourier series of F is summable A_2^+. The function u(x) above, restricted to $[-\pi,\pi)$ and then extended periodically, thus has u", its distributional second derivative, summable A_2^+ to 0 at 0. However u" is not Abel summable at 0 as a direct calculation shows so summability A_2^+ does not imply Abel summability.

7. <u>Mean Value Theorems for Generalized Riemann Derivatives</u>.

The prettiest type of mean value theorem would say something like this. Let $I = [x+b_0h, x+b_{n+e}h]$ where x and h are fixed. If $D_nf(t)$ exists for every $t \in I$, then there is a ξ interior to I with

$$\frac{\Delta_n(h;\mathbf{b},\mathbf{a})f(x)}{h^n} = D_nf(\xi).$$

But this is not even true for R_1 as the choices $x = -1$, $h = 3$ and $f(t) = |t|$ show.



I would suspect that the only generalized Riemann derivative for which this mean value theorem holds is $\frac{d}{dx}$ itself.

<u>Question 7</u>. Classify the D_n for which the mean value theorem in the above form is true.

A more fruitful set of mean value theorems are those of following type.

Statement $M(\mathbf{b},\mathbf{a})$. Fix x and h and set $I = [x + b_o h,\ x + b_{n+e} h]$. If $f^{(n-1)}(t)$ is continuous on I and differentiable for all t interior to I, then there is a ξ interior to I with $\dfrac{\Delta_n(h;\mathbf{b},\mathbf{a})f(x)}{h^n} = f^{(n)}(\xi)$.

A classification of the set of $(\mathbf{b},\mathbf{a})$ for which this statement is true is the goal of my present research with Roger Jones who is also at DePaul.[8]

We have a sufficient condition which is totally operational and which we can show to be necessary for all first and second generalized Riemann derivatives.

Let $p_o,\dots,p_e$ be real numbers with $\Sigma\ p_i=1$. Let $b_o< b_1 <\dots<b_{n+e}$ be n+1+e real numbers. Let D_o be the unique generalized n-th derivative based on $\{b_o,\dots,b_n\}$, D_1 the unique one based on $\{b_1,\dots,b_{n+1}\},\dots,$ D_e the unique one based on $\{b_e,\dots,b_{n+e}\}$, and set $D = \sum_{i=o}^{e} p_i D_i$. Then a quick check of the consistency condition shows that D is also an n-th derivative. Conversely, given any n-th generalized Riemann derivative D based on $\{b_o,\dots,b_{n+e}\}$ we can write D as $\Sigma\ p_i D_i$ where the p_i are uniquely determined by $\mathbf{b}$ and $\mathbf{a}$. The p_i are very easily found and satisfy $\Sigma\ p_i = 1$.

For example, O'Connor's derivative is associated to

$$\frac{7f(x+3h)-13f(x+4h)+6f(x+(16/3)h)}{h} =$$

$$7\left[\frac{f(x+3h)-f(x+4h)}{h}\right] - 6\left[\frac{f(x+4h)-f(x+(16/3)h)}{h}\right] =$$

$$-7\left[\frac{-f(x+3h)+f(x+4h)}{h}\right] + 8\left[\frac{-f(x+4h)+f(x+(16/3)h)}{(4/3)h}\right].$$

So letting D_0 and D_1 be the limits of the last 2 bracketed expressions, as $h \to 0$ we have $O_2 = p_0 D_0 + p_1 D_1$, where $p_o + p_1 = -7 + 8 = 1$.

<u>Theorem</u>. Let $D_n(\mathbf{b},\mathbf{a})$ be an n-th generalized Riemann derivative.

i) If the p_i associated to D are all positive (so that D is a convex combination of n-th derivatives without excess), then Theorem $M(\mathbf{b},\mathbf{a})$ holds.

ii) Conversely if n=1 or n=2 or e=1, and if any p_i is negative; then Statement $M(\mathbf{b},\mathbf{a})$ is false.

<u>Question 8</u>. What happens if $n \geq 3$, $e \geq 2$, and some p_i is negative? In particular, what happens for the excess 2 third derivative $D := (5/8)D_0 - (1/4)D_1 + (5/8)D_2$, where for i = 0, 1, 2,

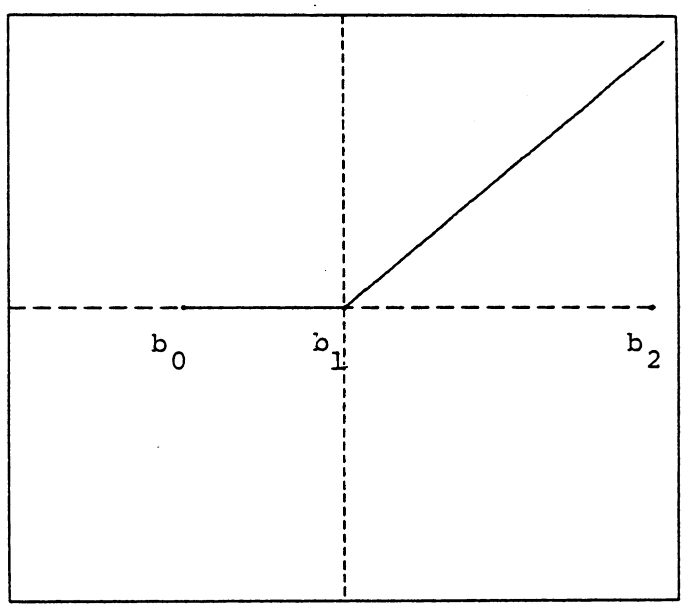
$D_i := -f(x+ih)+3f(x+[i+1]h)-3f(x+[i+2]h)+f(x+[i+3]h)$?

The proof of i) is short and sweet. First if e=0 then $p_o=1$ and indeed Theorem M is a well established numerical analysis fact.[12] If e>0, using this fact e+1 times we have numbers ξ_i so that

$$s = \frac{\Delta_n(h;\mathbf{b},\mathbf{a})f(x)}{h^n} = \sum_{i=o}^{e} p_i f^{(n)}(\xi_i).$$

The right side is a convex combination of the numbers $\{f^{(n)}(\xi_o),\ldots,f^{(n)}(\xi_e)\}$ and hence s lies between the smallest and the largest. But $f^{(n)} = (f^{(n-1)})'$ is an ordinary first derivative, hence is Darboux and therefore assumes the value s.

The proof of ii) is longer so we will restrict ourselves to one simple case. Let $b_o < b_1 < b_2$, let Δ_0 be the difference quotient associated to the unique first derivative based on $\{b_o,b_1\}$, Δ_1 the one based on $\{b_1,b_2\}$, and $\Delta = -7\Delta_o + 8\Delta_1$. Let f be this piecewise linear function.



Then $\Delta_1 = 1$, $\Delta_0 = 0$ so $\Delta=8$, but $f'=0$ or 1. Finally round the corner at b_1 very slightly. This will make Range(f')=[0,1] but keep Δ close to 8 so that the mean values theorem fails for Δ.

We do the second derivative case by piecing together quadratics and then rounding the corners. The example for the general n, excess 2 derivative case uses an nth degree polynomial.

REFERENCES

1. Ash, J.M., Generalizations of the Riemann derivative, Trans. Amer. Math. Soc. 126(1967), 181-199.

2. Ash, J.M., A characterization of the Peano derivative, Trans. Amer. Math. Soc. 149(1970), 489-501.

3. Ash, J.M., Ed, Studies in Harmonic Analysis, MAA Studies in Math., vol. 13.

4. Ash, J.M., A "new" proof of uniqueness for trigonometric series, unpublished manuscript, 1977, 1-3.

5. Ash, J.M., Erdos, P., and Rubel, L.A., Very slowly varying functions, Acquationes Math. 10(1974), 1-9.

6. Ash, J.M. and Welland, G., Convergence, uniqueness, and summability of multiple trigonometric series, Trans. Amer. Math. Soc. 163(1972), 401-436.

7. Ash, J.M. and Jones, R.L., Optimal numerical differentiation using three function evaluations, Math. Comp. 37(1981), 159-167.

8. Ash, J.M. and Jones, R.L., Mean value theorems for generalized Riemann derivatives, preprint.

9. Ash, J.M., Janson, S., and Jones, R.L., Optimal numerical differentiation using n function evaluations, Estrato da Calcolo, 21(1984), 151-169.

10. Cantor, G., Gesammelte Abhandlungen, Georg Olms, Hildesheim, 1962, 80-83. Beweis, das eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen läst, Crelles J.F. Math., 72(1870), 139-142.

11. Denjoy, A., Sur l'intégration des coefficients différentiels d'ordre supérieur, Fund. Math. 25(1935), 273-326.

12. Isaacson, E. and Keller, H.B., Analysis of Numerical Methods, John Wiley, New York, 1966.

13. Kuttner, B., The relation between Riemann and Cesaro Summability, Proc. London Math. Soc. 38(1935) 273-283.

14. O'Connor, P.J., Generalized differentiation of functions of a real variable, Ph.D. dissertation, Wesleyan Univ., Middletown, Conn., 1969.

15. Oliver, H.W., The exact Peano derivative, Trans. Amer. Math. Soc. 76(1954) 444-456.

16. Riemann, B., Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe, Ges. Werke, 2. Aufl., Leipzig, 1892, pps. 227-71. Also Dover, New York, 1953.

17. Zygmund, A., Trigonometric Series, Vol. I and II, Cambridge Univ. Press, Cambridge, 1959.