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A NEW PROOF OF FLEISSNER'S THEOREM ON PRODUCTS OF DERIVATIVES

Let I be an interval of the real line. Saying that $f: I \to \mathbb{R}$ is a derivative means there exists a differentiable function $F: I \to \mathbb{R}$ such that $F'(x) = f(x)$, $x \in I$. In this case, F will be called a primitive of f.

Let $\Delta = \{f : I \to \mathbb{R} \mid f$ is a derivative$\}$ and $B = \{g : I \to \mathbb{R} \mid f \cdot g \in \Delta$ for every $f \in \Delta\}$.

In [1] R.J. Fleissner using the theory of the Denjoy integral showed that every continuous function which is of bounded variation belongs to B. In the present paper we shall give a simple proof of Fleissner's result, which does not involve the theory of the Denjoy integral. The following result is well-known and will allow us a local treatment of the problem.

Theorem 1. If $(D_j)_{j \in J}$ is a covering of the interval I with open intervals and $f: I \to \mathbb{R}$ is a function with the property that the restriction of f to every interval D_j is a derivative, then $f: I \to \mathbb{R}$ is a derivative.

The proof of the following theorem is new.

Theorem 2 (Fleissner [1]). Let $f: I \to \mathbb{R}$ be a continuous map which is locally of bounded variation and let g be a derivative. Then the map $h = f \cdot g$ is a derivative.

Proof. In view of Theorem 1 without loss of generality we may suppose that f is a strictly increasing map on I. We shall prove that the map $H = f \cdot g - v \circ f$ is a primitive of h, where G is a primitive of g and v is a primitive of the map $u = G \circ f^{-1}$ on $f(I)$.

Let $a \in I$ and $x \neq a$. Then

$$\frac{H(x)-H(a)}{x-a} = \frac{f(x)G(x)-f(a)G(a)-v(f(x))+v(f(a))}{x-a} =$$

$$= f(x) \cdot \frac{G(x)-G(a)}{x-a} + \frac{(f(x)-f(a))G(a)-(v(f(x))-v(f(a)))}{x-a}.$$

We shall prove that the map

$$u(x) = \frac{(f(x) - f(a))G(a) - (v(f(x)) - v(f(a)))}{x-a} \qquad x \neq a$$

satisfies the condition

$$(1) \qquad \lim_{x \to a} u(x) = 0.$$

From the Lagrange mean value theorem it follows that for every $x \neq a$ there exists a real number $c(x)$ between $f(x)$ and $f(a)$ such that the following equality holds:

$$(2) \qquad v(f(x)) - v(f(a)) = v'(c(x))(f(x) - f(a)).$$

It follows that for every $x \neq a$ there exists a number $d(x)$ between x and a such that $f(d(x)) = c(x)$. From (2) we obtain:

$$v(f(x)) - v(f(a)) = u(c(x))(f(x) - f(a)) = G(d(x))(f(x) - f(a)).$$

Thus

$$u(x) = (f(x) - f(a)) \frac{G(a) - G(d(x))}{x-a}$$

Since $0 < |a - d(x)| < |x - a|$, we have

$$|u(x)| \leq |f(x) - f(a)| \cdot \left| \frac{G(d(x)) - G(a)}{d(x)-a} \right|$$

which proves (1). Hence

$$\lim_{x \to a} \frac{H(x)-H(a)}{x-a} = \lim_{x \to a} f(x) \frac{G(x) - G(a)}{x-a} + \lim_{x \to a} u(x) = f(a)g(a).$$

Consequently, we have proved that $H' = h$.

REFERENCES

[1] R.J. Fleissner, On the product of derivatives, Fund. Math. 88(1975), 173-178.

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