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ON THE STRUCTURE OF SOME FUNCTION SPACE

In the present paper we shall deal with some spaces of functions $f : [0,1] \to R$ (R - the real line). If $f : [0,1] \to R$, then $C(f)(D(f))$ stands for the set of all points of continuity (discontinuity) of the function f. Further $\omega_f(x)$ denotes the oscillation of f at x. (See [5], p. 120.) If $H \subset R$ is Lebesgue measurable, then $\lambda(H)$ stands for its Lebesgue measure.

Let $M(0,1)$ be the space of all bounded functions $f : [0,1] \to R$ furnished with the sup metric $d(f,g) = \sup\{|f(x) - g(x)|\}$ for $f,g \in M(0,1)$.

The following general result gives a description of the structure of some function spaces. (See Corollaries I - V.)

Theorem. _Let F be a linear space of bounded functions $f : [0,1] \to R$ furnished with the sup metric d. Suppose F satisfies_

$$(P) \qquad \forall_{t<1} \ \exists_{h=h_t \in F} \ \lambda(D(h)) > t \ .$$

Then the set $F^* = \{f \in F : \lambda(D(f)) = 1\}$ is a residual, G_δ set in (F,d).

Remark 1. If F contains a function h such that $\lambda(D(h)) = 1$, then F fulfills the condition (P) of the Theorem. In case residual subsets of F can be empty it can be asked if the condition (P) of the Theorem implies the existence of a function $h \in F$ such that $\lambda(D(h)) = 1$. The next example gives a negative answer to this question.

Example. Let F_0 be the class of all functions $f : [0,1] \to R$ such that f is a linear combination of characteristic functions of perfect, nowhere dense sets $A \subset [0,1]$. Obviously F_0 is a linear space. Since for each $t < 1$ there exists a perfect, nowhere dense set $A \subset [0,1]$ such that $\lambda(A) > t$, the condition (P) of the Theorem is fulfilled for F_0. Clearly there is no h in F_0 with the property $\lambda(D(h)) = 1$.

The proof of the above the Theorem requires two lemmas which we present now.

Lemma 1. Let $0 \leq \alpha < 1$ and let $F_\alpha = \{f \in F : \lambda(D(f)) > \alpha\}$. Then F_α is an open set in (F,d).

Proof. Let $f \in F_\alpha$. We construct $\epsilon > 0$ such that $K(f,\epsilon) = \{g \in F : d(f,g) < \epsilon\} \subset F_\alpha$. Define sets $D_n(f) \subset [0,1]$ in the following way: $D_1(f) = \{x \in [0,1] : \omega_f(x) \geq 1\}$, $D_n(f) = \{x \in [0,1] : n^{-1} \leq \omega_f(x) < (n-1)^{-1}\}$ $(n = 2,3,\cdots)$. Then $D(f) = \cup_{n=1}^{\infty} D_n(f)$ and each of the sets $D_n(f)$ is measurable ([5], p. 120). Since the sets $D_n(f)$ are mutually disjoint, we have $\lambda(D(f)) = \Sigma_{n=1}^{\infty} \lambda(D_n(f))$. It follows from $\lambda(D(f)) > \alpha$ that there is N such that

$$(1) \qquad \sum_{n=1}^{N} \lambda(D_n(f)) > \alpha .$$

Obviously $\omega_f(x) \geq N^{-1}$ for each $x \in \cup_{n=1}^{N} D_n(f)$. Using the definition of the oscillation $\omega_f(x)$ of f at x we have

$$(2 \qquad \forall_{\delta>0} \; \exists_{x_1,x_2\in(x-\delta,x+\delta)} \quad |f(x_1) - f(x_2)| > 3(4N)^{-1} .$$

For each $g \in K(f,(4N)^{-1})$ it follows from (2) that

$$(3) \qquad \forall_{\delta>0} \; \exists_{x_1,x_2\in(x-\delta,x+\delta)} \quad |g(x_1) - g(x_2)| \geq$$

$$|f(x_1) - f(x_2)| - |f(x_1) - g(x_1)| - |f(x_2) - g(x_2)| >$$

$$3(4N)^{-1} - (4N)^{-1} - (4N)^{-1} = (4N)^{-1} .$$

Consequently $x \in D(g)$ and we have

$$(4) \qquad \bigcup_{n=1}^{N} D_n(f) \subset D(g) .$$

It follows from (4) and (1) that $\lambda(D(g)) > \alpha$. Hence $K(f,(4N)^{-1}) \subset F_\alpha$. Each point of F_α is an interior point. Hence F_α is an open set in (F,d).

Lemma 2. Each of the sets F_α $(0 \leq \alpha < 1)$ is dense in (F,d).

Proof. Let $0 \leq \alpha < 1$, $f \in F$, $\epsilon_0 > 0$. We prove $F_\alpha \cap K(f,\epsilon_0) \neq \emptyset$. Let $D_n(f)$ $(n = 1,2,\cdots)$ be the sets introduced in the proof of Lemma 1. Then $\lambda(C(f)) + \sum_{n=1}^{\infty} \lambda(D_n(f)) = 1 > \alpha$. Hence there is an N such that

$$(5) \qquad \lambda(C(f)) + \sum_{n=1}^{N} \lambda(D_n(f)) > \alpha .$$

Put $\epsilon = \min\{\epsilon_0,(4N)^{-1}\}$ and take $t > 0$ such that $t < \lambda(C(f)) + \sum_{n=1}^{N} \lambda(D_n(f)) - \alpha$. It follows from the condition (P) of the Theorem that there exists a function $h = h_{1-t} \in F$ with $\lambda(D(h)) > 1 - t$. Choose $c > 0$ so small that $c\cdot\sup\{|h(x)|\} < \epsilon$. Put $g = f + c\cdot h$. Since F is a linear space, g belongs to F. Obviously $g \in K(f,\epsilon) \subset K(f,\epsilon_0)$. We prove the inclusion

$$(6) \qquad (C(f) \cap D(h)) \cup \bigcup_{n=1}^{N} D_n(f) \subset D(g) .$$

If $x \in C(f) \cap D(h)$, then there are sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$, $y_n \to x$ and $a = \lim_n h(x_n) \neq \lim_n h(y_n) = b$. Obviously $g(x_n) \to f(x) + ca$, $g(y_n) \to f(x) + cb$. Hence $x \in D(g)$. The inclusion $\bigcup_{n=1}^{N} D_n(f) \subset D(g)$ can be verified by the method of Lemma 1 used for the function $g \in K(f,(4N)^{-1})$.

The function h fulfills $\lambda(D(h)) > 1 - t$ and hence we have $\lambda(C(f) \cap D(h)) > \lambda(C(f)) - t$. It follows from (6), (5) and from the choice of t that $\lambda(D(g)) \geq \lambda(C(f)) - t + \sum_{n=1}^{N} \lambda(D_n(f)) > \alpha$. Hence $g \in F_\alpha$ and $g \in F_\alpha \cap K(f,\epsilon_0)$.

Proof of Theorem. Put $\alpha(n) = n(n + 1)^{-1}$ $(n = 1,2,\cdots)$. Then $F^* = \bigcap_{n=1}^{\infty} F_{\alpha(n)}$. According to Lemma 1 and Lemma 2 each of the sets $F_{\alpha(n)}$ is open and dense in (F,d). Consequently F^* is a residual, G_δ set in (F,d).

Remark 2. It follows from the Theorem that the space F_0 from Example is a metric space of the first category in itself.

We shall use the Theorem to describe the structure of some known function spaces. Note, that the following Corollaries I and II are contained in the paper [2].

Corollary I. _Let_ $b\Delta$ _be the space of all bounded derivatives on_ $[0,1]$ _with the sup metric. Then the class of all_ $b\Delta$ _functions_ f _such that_ $\lambda(D(f)) = 1$ _is a residual,_ G_δ _set of the second category in_ $b\Delta$.

Proof. Clearly $b\Delta$ is a linear space. It follows from the Theorem 6.5 of [1], p. 28 that there is a function $h \in b\Delta$ such that $\lambda(D(h)) = 1$. (See Remark 1.) Since $b\Delta$ is a complete metric space, ([1], pp. 17, 33), according to the Baire Category the Theorem ([5], p. 321) $b\Delta$ is a set of the second category in itself. Hence, the Theorem implies the assertion of Corollary I.

Corollary II. _Let_ bA _be the space of all bounded, approximately continuous functions on_ $[0,1]$ _with the sup metric. Then the class of all_ bA _functions_ f _such that_ $\lambda(D(f)) = 1$ _is a residual,_ G_δ _set of the second category in_ bA.

Proof. Since the space bA is complete ([1], the Theorem 5.7, p. 24), bA is of the second category in bA. Obviously bA is a linear space and the Theorem 2.3 of [1], p. 48 implies that there exists an $h \in bA$ such that $\lambda(D(h)) = 1$. The assertion of Corollary II is a consequence of the Theorem.

In the paper [3] (the Theorem 11) the following results (a), (b) and (c) are proved. Further P denotes either of above spaces $b\Delta$ and bA.

(a) The class of all $f \in P$ such that $\lambda(\overline{f(C(f))}) = 0$ is a residual G_δ set in P.

(b) The class of all $f \in P$ such that $f(C(f))$ is nowhere dense is a residual G_δ set in P.

(c) The class of all $f \in P$ such that $f(C(f))$ is nowhere dense and null is residual in P.

Combining the above results with Corollaries I and II we have

Corollary III. (i) *The class of all* $f \in P$ *such that* $\lambda(C(f)) = \lambda(\overline{f(C(f))}) = 0$ *is a residual* G_δ *set of the second category in* P.

(ii) *The class of all* $f \in P$ *such that* $\lambda(C(f)) = 0$ *and* $f(C(f))$ *is nowhere dense is a residual* G_δ *set of the second category in* P.

(iii) *The class of all* $f \in P$ *such that* $\lambda(C(f)) = \lambda(\overline{f(C(f))}) = 0$ *and* $f(C(f))$ *is nowhere dense is a residual set of the second category in* P.

Recall that a function $f : [0,1] \to R$ is said to have the Baire property if there is a set $B = B_f$, $B \subset [0,1]$, of the first category in $[0,1]$ such that $f|([0,1] - B)$ is a continuous function ([5], p. 190).

Corollary IV. *Let* Q *be the space of all bounded functions* $f : [0,1] \to R$ *with the Baire property endowed with the sup metric. Then the class of all* $f \in Q$ *such that* $\lambda(D(f)) = 1$ *is a residual,* G_δ *set of the first category in* Q.

Proof. Obviously Q is a linear space and $Q \subset M(0,1)$ is a closed subset of $M(0,1)$. Hence Q is a complete metric space and Q is a set of the second category in Q. The condition (P) of the Theorem is fulfilled for Q since the Dirichlet function h belongs to Q and $\lambda(D(h)) = 1$. The assertion is a consequence of the Theorem.

Corollary V. *Let* Q^* *be the space (with the sup metric) of all bounded functions* $f : [0,1] \to R$ *for which* $C(f)$ *is a residual set. Then the class of all* $f \in Q^*$ *such that* $\lambda(D(f)) = 1$ *is a residual,* G_δ *set of the second category in* Q.

Proof. Q^* is a linear space. Since Q^* is a closed subset of $M(0,1)$, (Q^*,d) is complete. Let $L \subset [0,1]$ be a null and residual G_δ set in $[0,1]$. (The set of all Liouville numbers of $[0,1]$ has this property. See [4], pp. 8-10.) Let $h : [0,1] \to R$ be a bounded function continuous on L and discontinuous on $[0,1] - L$. Then $\lambda(D(h)) = 1$, $h \in Q^*$ and the assertion follows from the Theorem.

References

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