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BLACKWELL SPACES AND GENERALISED LUSIN SETS

§0. Introduction

By a separable space is meant a measurable (Borel) space $(X, \mathcal{B})$ whose σ-algebra $\mathcal{B}$ is countably generated (c.g.) and contains all singletons drawn from X. If $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}$, and $A \subset X$, then the notation $\mathcal{C}(A) = \{B \cap A : B \in \mathcal{C}\}$ will be used to denote the relative $\mathcal{C}$-structure on A. The separable space $(S, \mathcal{B})$ is standard (resp. analytic) if there is a complete (resp. analytic) separable metric topology on S for which $\mathcal{B}$ is the corresponding Borel structure.

A separable space $(X, \mathcal{B})$ has the Blackwell property if whenever $\mathcal{C}$ is a c.g. sub-σ-algebra of $\mathcal{B}$ separating points of X, then perforce $\mathcal{C} = \mathcal{B}$. Say that $(X, \mathcal{B})$ has the strong Blackwell property if whenever $\mathcal{C} \subset \mathcal{D}$ are c.g. sub-σ-algebras of $\mathcal{B}$ with the same atoms, then necessarily $\mathcal{C} = \mathcal{D}$.

A reasonably extensive survey of results about the Blackwell properties is [1]. We content ourselves with a brief review.

Fact 1: Let $(X, \mathcal{B})$ be a separable space. Then the following statements are equivalent:

1) $(X, \mathcal{B})$ has the Blackwell property;

2) whenever f is a one-one measurable real-valued function on $(X, \mathcal{B})$, then f is a Borel isomorphism of X onto its image $f(X)$.

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Fact 2: Let $(X, \mathcal{B})$ be a separable space. Then the following are equivalent:

1) $(X, \mathcal{B})$ has the strong Blackwell property;

2) whenever f is a measurable real function on $(X, \mathcal{B})$, then $f(X) \subset \mathbb{R}$ has the Blackwell property.

Fact 3: Every analytic space has the strong Blackwell property.

Fact 4: There is at least one co-analytic space without the Blackwell property.

We shall obtain fact 3 as a corollary of our general theory: see proposition 5 *infra*. It is also worth mentioning that there are some rather pathological (not universally measurable) spaces with the strong Blackwell property.

Proofs of all of the above facts and a short history of the subject are to be found in the monograph [1]. The question of whether there is a Blackwell space without the strong Blackwell property is unsettled except under some extra set-theoretic assumptions: this according to some unpublished work of D. Fremlin, W. Bzyl and J. Jasiński.

At the 1983 Oberwolfach Conference on Measure Theory, the first author posed the problem of whether the construction of universally null Blackwell spaces is possible within ZFC. At present, the matter does not seem to have found resolution. However, Jakub Jasiński [4] in Gdańsk has very recently shown that under the assumption of CH, there are some Lusin sets with the Blackwell property and some without. As he points out, his arguments apply also to their category analogue, the Sierpiński sets.

The present paper is an effort to unify these results with other theorems about Blackwell spaces in a rather general framework. Many examples of Blackwell spaces may then be seen to arise as generalised Lusin sets with respect to a σ-ideal of Borel sets. Vary the σ-ideal, and different species of Blackwell spaces emerge. Additionally, our method gives a way of characterising such spaces, at least when the σ-ideal is "uniformisable".

A preliminary section introduces the basic ideas, including the notion of uniformisability and degree of density with respect to a σ-ideal. The main results, propositions 1 and 2, characterise generalised Lusin sets with the Blackwell property and prove that for these sets, Blackwell and strong Blackwell properties coincide. Applications to four types of σ-ideal are given: countable sets, sets of measure zero, sets of first category, and a special type of σ-ideal of sets avoiding a given set.

There are some important open questions: we do not know whether the σ-ideals for measure and category are uniformisable. A result along these lines would involve some new theorem on measurable selections.

§1. Preliminaries

We assume that the reader is familiar with the elements of descriptive set theory and Borel structures as presented in [1] and [5]. Our notation and terminology conform in large degree to the paradigms in [1].

Until further notice, $(S, \mathcal{B})$ will denote a fixed uncountable standard space. A (proper) σ-ideal $\mathcal{I}$ in $\mathcal{B}(S)$ is <u>continuous</u> if it contains all singletons drawn from S. A subset X of S is <u>$\mathcal{I}$-Lusin</u> if X is uncountable, and the intersection of X with every member of $\mathcal{I}$ is countable. A subset R of $S \times S$ is <u>$\mathcal{I}$-reticulate</u> if there is a set N in $\mathcal{I}$ such, that $R \subset (N \times S) \cup (S \times N)$. A subset T of $S \times S$ is an <u>$\mathcal{I}$-thread</u> if

1) T is the graph of a Borel-isomorphism between sets A and B in $\mathcal{B}(S)$;

2) T is not $\mathcal{I}$-reticulate.

A σ-ideal $\mathcal{I}$ is uniformisable if every set R in $\mathcal{B}(S \times S)$ which is not $\mathcal{I}$-reticulate contains an $\mathcal{I}$-thread. A subset X of S is $\mathcal{I}$-dense (of order 1) if every $B \in \mathcal{B}(S)$ such that $X \cap B = \emptyset$ is a member of $\mathcal{I}$. Say that X is $\mathcal{I}$-dense of order 2 if $X \times X$ meets every R in $\mathcal{B}(S \times S)$ which is not $\mathcal{I}$-reticulate. It is not hard to see that $\mathcal{I}$-density of order 2 implies $\mathcal{I}$-density of order 1.

The explicit designation of $\mathcal{I}$ in the terms "$\mathcal{I}$-Lusin", "$\mathcal{I}$-reticulate" &c. will occasionally be suppressed.

Suppose now that $\mathcal{C}$ and $\mathcal{D}$ are c.g. sub-σ-algebras of $\mathcal{B}(S)$. Say that $\mathcal{C}$ is proper in $\mathcal{D}$ ("$\mathcal{I}$-proper") if

1) $\mathcal{C} \subset \mathcal{D}$, and

2) if $N \in \mathcal{I}$, then $\mathcal{C}(N^c) \neq \mathcal{D}(N^c)$.

Clearly, $\mathcal{C}$ is proper in $\mathcal{D}$ if and only if there is a set D in $\mathcal{D}$ not equivalent to any $\mathcal{C}$-set modulo $\mathcal{I}$, i.e. $D \,\Delta\, C \notin \mathcal{I}$ for any C in $\mathcal{C}$.

If $\mathcal{C}$ is any c.g. sub-σ-algebra of $\mathcal{B}(S)$, there is some real-valued function f defined on S such that $\mathcal{C} = \mathcal{B}_f = \{f^{-1}(B) : B \subset \mathbb{R} \text{ Borel}\}$. We call any such function f a Marczewski function for $\mathcal{C}$. If C_1, C_2, ... is a sequence of sets generating $\mathcal{C}$, then f may be defined as

$$f(x) = \sum_{n=1}^{\infty} 2 \cdot I_{C_n}(x)/3^n ,$$

where I_C is the indicator (characteristic) function of C.

Given Marczewski functions f and g for any c.g. $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C} \subset \mathcal{D}$, we define the set

$$T(\mathcal{C}, \mathcal{D}) = \{(s, t) \in S \times S : f(s) = f(t) \text{ and } g(s) \neq g(t)\}$$

The definition of $T(\mathcal{C}, \mathcal{D})$ does not depend on the particular choice of the Marczewski functions f and g. In the same context, we shall say that a subset X of S satisfies condition (J) or (J+) if:

(J) whenever $\mathcal{C}$ is a c.g. sub-σ-algebra of $\mathcal{B}(S)$ which is proper in $\mathcal{B}(S)$, then there is some $\mathcal{C}$-atom C such, that $C \cap X$ contains at least two points (i.e. $\mathcal{C}$ does not separate points of X);

(J+) whenever $\mathcal{C}$ and $\mathcal{D}$ are c.g. sub-σ-algebras of $\mathcal{B}(S)$ with $\mathcal{C}$ proper in $\mathcal{D}$, then there is some $\mathcal{C}$-atom C such, that $C \cap X$ contains two points separated by $\mathcal{D}$.

Lemma 1: Let $\mathcal{I}$ be a uniformisable, continuous σ-ideal. Then condition (J) implies that X is $\mathcal{I}$-dense.

Proof: If X is not dense in S, then there is some set $B \subset S \setminus X$ in $\mathcal{B} \setminus \mathcal{I}$. Since $\mathcal{I}$ is continuous, B decomposes into two (necessarily uncountable) disjoint set B_1 and B_2 in $\mathcal{B} \setminus \mathcal{I}$. The set $B_1 \times B_2$ is not $\mathcal{I}$-reticulate. By the uniformisability of $\mathcal{I}$, the set $B_1 \times B_2$ contains an $\mathcal{I}$-thread G. Now G is the graph of a Borel-isomorphism g between sets C_1 and C_2 in $\mathcal{B}(S)$. Define $f : S \to S$ by the rule

$$f(s) = \begin{cases} g(s) & s \in C_1 \\ s & s \notin C_1 \end{cases}$$

We claim that $\mathcal{C} = \mathcal{B}_f$ is proper in $\mathcal{B}(S)$. For each $N \in \mathcal{I}$, there is in G a point $(s, g(s))$ not in $(N \times S) \cup (S \times N)$. Then s and $g(s)$ are points of N^c not separated by $\mathcal{C}$. So $\mathcal{C}(N^c) \neq \mathcal{B}(N^c)$.

However, $\mathcal{C}$ separates points of X. The lemma follows by contraposition.

Q.E.D.

Lemma 2: If X is an $\mathcal{I}$-Lusin subset of S, then the following lattice of implications obtains:

$$\begin{array}{ccc} X \text{ satisfies } (J+) & \longrightarrow & X \text{ is strongly Blackwell} \\ \downarrow & & \downarrow \\ X \text{ satisfies } (J) & \longrightarrow & X \text{ is Blackwell .} \end{array}$$

<u>Proof:</u> We show that if X satisfies condition (J+), then X is strongly Blackwell. The other "horizontal" implication runs entirely parallel, whilst the "vertical" ones are trivial. So suppose that $\mathcal{C}(X) \subset \mathcal{D}(X)$ are c.g. sub-σ-algebras of $\mathcal{B}(X)$ with the same atoms. Then there are c.g. σ-algebras $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}(S)$ whose relativisations to X are $\mathcal{C}(X)$ and $\mathcal{D}(X)$. Condition (J+) implies that $\mathcal{C}$ is not proper in $\mathcal{D}$. Therefore, there is some set N in $\mathcal{I}$ with $\mathcal{C}(N^c) = \mathcal{D}(N^c)$. Since X is an $\mathcal{I}$-Lusin set, it follows that $X \cap N$ is countable.

Let A be the union of all $\mathcal{C}$-atoms C with the property that $C \cap X \cap N \neq \emptyset$. There are only countably many such $\mathcal{C}$-atoms, so that $A \in \mathcal{C}(S)$. Since $\mathcal{C}(X \cap A)$ and $\mathcal{D}(X \cap A)$ have the same countable set of atoms, one has $\mathcal{C}(X \cap A) = \mathcal{D}(X \cap A)$. Now $X \cap A^c \cap N^c$, so that $\mathcal{C}(X \cap A^c) = \mathcal{D}(X \cap A^c)$. Given D in $\mathcal{D}$, there are $\mathcal{C}$-sets C_1 and C_2 such that

$$\begin{aligned} D \cap X &= (D \cap X \cap A) \cup (D \cap X \cap A^c) \\ &= (C_1 \cap X \cap A) \cup (C_2 \cap X \cap A^c) \end{aligned}$$

is a member of $\mathcal{C}(X)$. So $\mathcal{C}(X) = \mathcal{D}(X)$.

Q.E.D.

§2. The Main Results.

Proposition 1: Suppose $\mathcal{I}$ is continuous and uniformisable. X be a subset of S . Then the following are equivalent:

1) X is $\mathcal{I}$-dense of order 2 in S ;

2) X satisfies condition (J+);

3) X satisfies condition (J) .

Remark: The implications 1. → 2. → 3. are true for general σ-ideals $\mathcal{I}$. Continuity and uniformisability are used in the proof that 3. implies 1. .

Demonstration: 1. implies 2.: Assume that $\mathcal{C} \subset \mathcal{D}$ are c.g. sub-σ-algebras of $\mathcal{B}(S)$ with $\mathcal{C}$ proper in $\mathcal{D}$. Let $T = T(\mathcal{C}, \mathcal{D})$ be as previously defined.

Claim: T is not $\mathcal{I}$-reticulate.

Suppose to the contrary that there is a set N in $\mathcal{I}$ with $T \subset (N \times S) \cup (S \times N)$. Since $\mathcal{C}$ is proper in $\mathcal{D}$, one has $\mathcal{C}(N^c) \neq \mathcal{D}(N^c)$. Since $N^c \in \mathcal{B}(S)$ is strongly Blackwell, it follows that there are points s , t in N^c separated by $\mathcal{D}$, but not by $\mathcal{C}$. Thus $(s, t) \in T$, a contradiction. The claim is established.

Since X is $\mathcal{I}$-dense of order 2, one has $(X \times X) \cap T \neq \emptyset$. So there are points x and x' in X separated by $\mathcal{D}$, but not by $\mathcal{C}$. Condition (J+) holds.

2. implies 3.: Trivial.

3. implies 1.: Here, we will assume that $S =]0, 1[$ and will employ the usual linear ordering on $]0, 1[$. This is justified since all uncountable standard spaces are Borel-isomorphic.

Assume that X is not $\mathcal{I}$-dense of order 2, so that there is some nonreticulate set R in $\mathcal{B}(S \times S)$ with $R \subset (S \times S)\setminus(X \times X)$. Since $\mathcal{I}$ is assumed uniformisable, R contains a thread G . Define

$$\Delta = \{(s, s) : s \in S\}$$

$$\Delta_+ = \{(s, t) \in S \times S : s < t\}$$

$$\Delta_- = \{(s, t) \in S \times S : s > t\} \quad .$$

From lemma 1, it follows that $G \cap \Delta$ is reticulate, and so we may assume without loss of generality that $G \cap \Delta_-$ is a thread. In fact, there is an $\varepsilon > 0$ such that $G \cap \Delta_-(\varepsilon)$ is a thread, where

$$\Delta_-(\varepsilon) = \{(s, t) \in S \times S : s - \varepsilon > t\} \; .$$

Also, there is some open interval $\mathcal{O}$ of length ε such that $G \cap \Delta_-(\varepsilon) \cap (\mathcal{O} \times S)$ is a thread. This set is the graph of a Borel isomorphism h defined on a Borel subset D of S .

Now, whenever s and t are elements of D , then $h(s) < s - \varepsilon < t$, so that $h(D) \cap D = \emptyset$. Define $g : S \to S$ by the rule

$$g(s) = \begin{cases} h(s) & \text{for } s \in D \\ s & \text{for } s \in D^c \; . \end{cases}$$

Since $G \cap (X \times X)$ is void, it follows that g is one-one on X .

Put $\mathcal{C} = \mathcal{B}g = \{g^{-1}(A) : A \in \mathcal{B}(S)\}$. We claim that $\mathcal{C}$ is proper in $\mathcal{B}(S)$. Suppose that $N \in \mathcal{I}$. Since $\text{graph}(h) = G \cap \Delta_-(\varepsilon) \cap (\mathcal{O} \times S)$ is not reticulate, there is some point $(s, h(s))$ not in $(N \times S) \cup (S \times N)$. Thus s and $h(s)$ are distinct points in N^c not separated by $\mathcal{C}$. So $\mathcal{C}$ is proper in $\mathcal{B}(S)$, yet $\mathcal{C}$ separates points of X : thus condition (J) fails.

Q.E.D.

Proposition 2: Suppose $\mathcal{I}$ is a continuous, uniformisable σ-ideal. Let X be an $\mathcal{I}$-Lusin set $\mathcal{I}$-dense in S . The following are equivalent:

1) X is $\mathcal{I}$-dense of order 2;

2) X is strongly Blackwell;

3) X is Blackwell.

Note: The implication 1. implies 2. does not require $\mathcal{I}$ to be uniformisable or X to be $\mathcal{I}$-dense; 3. implies 1. does not require $\mathcal{I}$ to be continuous or X to be $\mathcal{I}$-Lusin; 2. implies 3. needs no condition on $\mathcal{I}$ or X.

Demonstration: 1. implies 2.: This follows from proposition 1 and lemma 2.

2. implies 3.: Trivial.

3. implies 1.: We show that condition (J) is satisfied. Suppose that $\mathcal{C}$ is a c.g. sub-σ-algebra of $\mathcal{B}(S)$ with $\mathcal{C}(X)$ separable. Let f be a Marzcewski function for $\mathcal{C}$; if X is Blackwell, then $\mathcal{C}(X) = \mathcal{B}(X)$, and f is a Borel-isomorphism when restricted to X. From Kuratowski [5; p. 436], it follows that the restriction of f to X extends to a Borel-isomorphism g on a Borel set $B' \supset X$. Putting $B = B' \cap \{s : f(s) = g(s)\}$, we see that f is also a Borel-isomorphism on the Borel set $B \supset X$. Since X is dense, $S \backslash B \in \mathcal{I}$. This implies that no c.g. sub-σ-algebra can be proper and still separate points of X. Condition (J) obtains.

Q.E.D.

§3. Applications

Example 1: Let $\mathcal{I}$ be the σ-ideal of all countable subsets of the uncountable standard space S. Then $\mathcal{I}$ is continuous, and the $\mathcal{I}$-Lusin sets are precisely the uncountable subsets of S. The notions of $\mathcal{I}$-reticulate set, $\mathcal{I}$-thread, and $\mathcal{I}$-density coincide with those of "reticulate set", "thread" and "Borel-density" as used in [10], [11], and [12]. Results in [3], as well as an argument in Sarbadhikari's note [8] show that this σ-ideal is uniformisable.

Proposition 2 therefore applies to prove the following result (compare [11]):

Proposition 3: Let X be a subset of a Polish space S such that $S \setminus X$ is totally imperfect. Then the following are equivalent.

1) X is Blackwell;

2) X is strongly Blackwell;

3) X is Borel-dense of order 2.

Example 2: Let X be a fixed uncountable subset of X. Define $\mathcal{I}(X)$ to be the σ-ideal consisting of all B in $\mathcal{B}(S)$ with $B \cap X$ countable. Then $\mathcal{I}(X)$ is continuous, and X is $\mathcal{I}(X)$-dense of order 1.

Proposition 4: Let A be an analytic subset of S. Then A is $\mathcal{I}(A)$-dense of order 2 in S.

Demonstration: Let R be a member of $\mathcal{B}(S \times S)$ disjoint from $A \times A$. Then $R_1 = R \cap (S \times A)$ is an analytic set whose projection onto the first factor is disjoint from A. Lusin's first separation principle implies that there is some $N_1 \in \mathcal{B}(S)$ such that $N_1 \cap A = \emptyset$ and $R_1 \subset (N_1 \times S)$. Now $R_2 = R \setminus (N_1 \times S)$ is a member of $\mathcal{B}(S \times S)$ whose projection onto the second factor does not meet A. Thus there is some $N_2 \in \mathcal{B}(S)$ such that $N_2 \cap A = \emptyset$ and $R_2 \subset (S \times N_2)$. Then $N = N_1 \cup N_2 \in \mathcal{I}(A)$, and $R \subset (N \times S) \cup (S \times N)$.

Q.E.D.

Lemma 3: Let A be an analytic subset of S. The sets R in $\mathcal{B}(S \times S)$ which are not $\mathcal{I}(A)$-reticulate are precisely those for which $R \cap (A \times A)$ is not contained in a countable union of horizontal and vertical sections.

Proof: Clearly, any R with this property is not $\mathcal{I}(A)$-reticulate. Conversely, if $R \cap (A \times A)$ is contained in a countable union of sections, then by removing these sections from R, we obtain a Borel set R_o disjoint from

$A \times A$. From proposition 4, R_o is $\mathcal{I}(A)$-reticulate. So also is R .

Q.E.D.

Proposition 5: If A is analytic, then the σ-ideal $\mathcal{I}(A)$ is uniformisable.

Remark: This result, when combined with propositions 2 and 4, shows that every analytic space is strong Blackwell. A strengthening of this face will be proved in proposition 7.

Demonstration: Suppose that R is a member of $\mathcal{B}(S \times S)$ not $\mathcal{I}(A)$-reticulate. Then from lemma 3, $R \cap (A \times A)$ is not contained in a countable union of sections. It follows from [3; Theorem 4.4] or [8] that $R \cap (A \times A)$ contains an uncountable standard set T each of whose horizontal and vertical sections is at most a singleton. See also the discussion in [12] . Thus T is an $\mathcal{I}(A)$-thread.

Q.E.D.

If X is any uncountable subset of S , then certainly X is an $\mathcal{I}(X)$-Lusin set. However, there may be a great many other $\mathcal{I}(X)$-Lusin sets essentially larger than X .

Proposition 6: Let A be an analytic subset of S . Then the $\mathcal{I}(A)$-Lusin sets are those uncountable subsets Y of S whose intersection with each constituent of the co-analytic set $S\setminus A$ is countable.

Demonstration: If X is an $\mathcal{I}(A)$-Lusin set, and C is a constituent of $S\setminus A$, then $C \in \mathcal{I}(A)$, so that $X \cap C$ is countable. On the other hand, suppose X intersects each constituent of $S\setminus A$ in a countable set. Given N in $\mathcal{I}(A)$, we know that $N \cap A$ is countable, so that $N \cap A^c$ is Borel and contained in a countable union of constituents (boundedness theorem). This forces $X \cap N$ to be countable.

Q.E.D.

<u>Proposition 7</u>: Let A be an analytic subset of S and suppose that Y is a set whose intersection with each constituent of $S \backslash A$ is countable. Then $A \cup Y$ is strongly Blackwell.

<u>Demonstration</u>: From proposition 4, the sets A and $A \cup Y$ are $\mathcal{I}(A)$-dense of order 2. Proposition 6 says that $A \cup Y$ is $\mathcal{I}(A)$-Lusin. Proposition 1 ensures that $A \cup Y$ satisfies condition (J+). The result now follows from lemma 2.

Q.E.D.

In the case of non-analytic Blackwell spaces, a weaker form of proposition 7 is available:

<u>Proposition 8</u>: Let X be an uncountable Blackwell subset of S. If Y is an $\mathcal{I}(X)$-Lusin set, then $X \cup Y$ is also Blackwell.

<u>Demonstration</u>: The proof of proposition 2 (3 implies 1) shows that $X \cup Y$ satisfies condition (J) for $\mathcal{I} = \mathcal{I}(X)$. Clearly, $X \cup Y$ is $\mathcal{I}(X)$-Lusin so that lemma 2 applies to prove $X \cup Y$ Blackwell.

Q.E.D.

Define a <u>uniformisable set</u> to be a subset X of an uncountable standard space S for which $\mathcal{I}(X)$ is uniformisable. We have shown that every analytic set is uniformisable. If X is such that $S\backslash X$ is totally imperfect, then results in [3] and [8] show that X is a uniformisable set.

<u>Proposition 9</u>: Let X be a subset of S which is Blackwell, but not strongly Blackwell. Then X is not a uniformisable set.

<u>Demonstration</u>: As mentioned, X is certainly an $\mathcal{I}(X)$-Lusin set $\mathcal{I}(X)$-dense of order 1. The result is immediate from proposition 2.

Q.E.D.

In some unpublished work, D. Fremlin, W. Bzyl and J. Jasiński use axioms CH and (MA and not - CH) to construct Blackwell spaces without the strong Blackwell property. Whether such a space may be proved to exist in ZFC is not known.

<u>Conjecture 1</u>: The existence of a non-uniformisable set may be demonstrated in ZFC .

§4. <u>The construction of $\mathcal{I}$-Lusin sets</u>

The following result is patterned after Lusin's original construction [6] and the work of Jasiński [4] . In view of lemma 2, a variety of Blackwell spaces may now be produced, one for each σ-ideal on S .

<u>Proposition 10</u>: (CH) Suppose that $\mathcal{I}$ is a continuous σ-ideal in $\mathcal{B}(S)$. Then S contains an $\mathcal{I}$-Lusin set $\mathcal{I}$-dense of order 2.

<u>Demonstration</u>: List the members of $\mathcal{I}$ in transfinite series $I_0\ I_1\ I_2\ \ldots\ I_\alpha\ \ldots\ \alpha < \aleph_1$ and the non-$\mathcal{I}$-reticulate members of $\mathcal{B}(S \times S)$ as $R_0\ R_1\ \ldots\ R_\alpha\ \ldots\ \alpha < \aleph_1$. For each $\alpha < \aleph_1$, put $J_\alpha = \cup\ \{I_\beta : \beta \leq \alpha\}$. Choose points (x_α, y_α) from $R_\alpha \backslash (J_\alpha \times J_\alpha)$ for each $\alpha < \aleph_1$. Then the set $X = \{x_\alpha, y_\alpha : \alpha < \aleph_1\}$ intersects each I_α in a countable set, and $X \times X$ meets each R_α , as desired.

Q.E.D.

§5. Measure and Category

Example 3: Let m be a continuous probability measure on S and define $\mathcal{I} = \mathcal{I}(m)$ to be the collection of all m-null sets in $\mathcal{B}(S)$. Then $\mathcal{I}$ is continuous, and the $\mathcal{I}$-Lusin subsets of S are the classical Sierpiński sets [2]. A subset X of S is $\mathcal{I}$-dense if and only if $m^*(X) = 1$.

Conjecture 2: The σ-ideal $\mathcal{I}(m)$ is uniformisable.

To prove this conjecture would seem to require a new result in measurable selection theory. To facilitate a discussion of the conjecture, it becomes convenient to introduce the following set function, defined on subsets R of $S \times S$:

$$\mu R = \inf \{mA + mB : R \subset (A \times S) \cup (S \times B) \; ; \; A, B \in \mathcal{B}(S)\} .$$

Lemma 4: A set $R \subset S \times S$ is $\mathcal{I}(m)$-reticulate if and only if $\mu R = 0$.

Proof: One direction (only if) is obvious. If $\mu R = 0$, choose sets A_n, B_n in $\mathcal{B}(S)$ with $R \subset (A_n \times S) \cup (S \times B_n)$ and $mA_n + mB_n < 2^{-n}$. Put $A = \limsup A_n$ and $B = \limsup B_n$. Then $R \subset (A \times S) \cup (S \times B)$, and, from the Borel-Cantelli lemma, $mA = mB = 0$. So R is $\mathcal{I}(m)$-reticulate.

Q.E.D.

Lemma 5: Suppose that $(S \times S) \backslash R$ is a countable union of Borel rectangles (e.g. if S is metric and R is closed). Then $\mu R \geq \varepsilon$ if and only if there is a measure ν on $\mathcal{B}(S \times S)$ both of whose marginals equal m such that $\nu R \geq \varepsilon$.

Proof: This is essentially a result of Strassen [13], discussed and extended for the measurable setting in [9].

Q.E.D.

The $\mathcal{I}(m)$-threads are, roughly speaking, partial functions which do not change too many m-null sets. Conjecture 2 seems to be a statement that supports of "m-doubly stochastic" measures contain graphs of isomorphisms non-trivial for m .

Even without uniformisability of $\mathcal{I}(m)$, proposition 10 and lemma 2 imply that there are classical Sierpiński sets with the strong Blackwell property, at least under the aegis of CH . Jasiński [4] has constructed (CH) a Sierpiński subset X_1 of S such that if $X_2 \supset X_1$ is another Sierpinski set, then automatically X_2 is strong Blackwell. In his example, $m^*(X_1) = 1$. We see that if conjecture 2 holds, such behavior is actually quite typical.

<u>Proposition 11 (Conjecture 2)</u>: Suppose X_1 and X_2 are $\mathcal{I}(m)$-Lusin sets ("m-Sierpiński sets") with $X_1 \subset X_2$ and $m^*(X_1) = m^*(X_2)$. If X_1 is Blackwell, then X_2 is strongly Blackwell.

<u>Demonstration</u>: Choose $B \supset X_2$ with $B \in \mathcal{B}(S)$ and $mB = m^*(X_2)$. We know that $m^*(X_2) > 0$, so that $n = m/m(B)$ is a continuous probability on the standard space $(B, \mathcal{B}(B))$. Since X_1 is $\mathcal{I}(n)$-dense in B , proposition 2 implies that X_1 is $\mathcal{I}(n)$-dense of order 2 in B . So also is X_2 , and the result follows.

Q.E.D.

The following two results relate to the minimality and maximality of Sierpiński sets with and without the Blackwell property with respect to m-outer measure.

<u>Proposition 12 (CH and Conjecture 2)</u>: Let X_1 be an $\mathcal{I}(m)$-Lusin subset of S with the Blackwell property. Then there is an $\mathcal{I}(m)$-Lusin set $X_2 \subset X_1$ without the Blackwell property and such, that $m^*(X_2) = m^*(X_1)$.

<u>Demonstration</u>: By the usual isomorphism tricks, one may assume that $S =]0, 1[$ and that m is Lebesgue measure. Also, there is no loss of generality in assuming that $m^*(X_1) = 1$. Using CH , arrange the non-m-null members of $\mathcal{B}(S)$ in tranfinite series $B_o\ B_1 \ldots B_\alpha \ldots \alpha < \aleph_1$.

Choose points x_o , x_1 ,..., x_α ,... from X_1 by the rule $x_\alpha \in (B \cap X_1) \setminus \{1 - x_\beta : \beta < \alpha\} \setminus \{1/2\}$. Put $X_2 = \{x_o , x_1 ,...\}$. Then X_2 is $I(m)$-dense, but $X_2 \times X_2$ does not meet the $I(m)$-thread $y = 1 - x$ in $S \times S$. Therefore X_2 is not $I(m)$-dense of order 2 and so lacks the Blackwell property (proposition 2) .

Q.E.D.

Proposition 13 (CH): Let X_1 be an $I(m)$-Lusin subset of S without the Blackwell property. Then there is an $I(m)$-Lusin set $X_2 \supset X_1$ with the Blackwell property and such, that $m^*(X_1) = m^*(X_2)$.

Demonstration: Choose a set B in $\mathcal{B}(S)$ with $X_1 \subset B$ and $m^*(B) = m(B)$. Under CH , use proposition 10 to obtain an $I(m)$-Lusin subset Y of B $I(m)$-dense in B of order 2. Then $X_2 = X_1 \cup Y$ is also such a set, and is by lemma 2 strongly Blackwell. Clearly, $m^*(X_2) = m^*(X_1)$.

Q.E.D.

Since the existence of $I(m)$-Lusin sets is presumed in proposition 12 and 13, it may be that CH can be relaxed for the proofs.

Example 4: Let τ be a Polish topology on S generating the Borel structure $\mathcal{B}(S)$. Define $I = I(\tau)$ to be the collection of all τ-first category sets in $\mathcal{B}(S)$. Then I is continuous if and only if τ has no isolated points, which we will assume. Then the I-Lusin subsets of S are the classical Lusin sets [2] .

Conjecture 3: The σ-ideal $I(\tau)$ is uniformisable. As in the case of conjecture 2, a proof would involve some new result in measurable selection theory. Although similar findings have been reported ([3], [7]), they do not seem to apply here.

Assuming conjecture 3, it becomes possible to prove analogues to propositions 11-13. Given the unsettled position of the conjecture, it seems wise to leave their formulation to the reader's fancy.

Added in proof: Results along the lines of proposition 8 have been obtained by Jakub Jasiński (unpublished).

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