Real Analysis Exchange Vol. 10 (1984-85)

Gabriela Ene, Institute of Mathematics, str. Academiei 14, 70109 Bucharest, Romania

An Extension of the Ordinary Variation

In [2] Foran has introduced condition $B(N)$ which for $N = 1$ is identical to the condition of bounded variation. Using condition $B(N)$ we introduce the variation $V_N(F;E)$ of a function F on a set E which for $N = 1$ is identical to the ordinary variation of F on E. Then we show that there exist functions F on $[0,1]$ for which $V_2(F;[0,x] \cap C) = \phi(x)$ (C = Cantor's ternary set, ϕ = Cantor's ternary function, $x \in C$). Using this new variation we show that there exist continuous functions, satisfying Lusin's condition (N) on $[0,1]$, which are $B(N)$ on C for no natural number N.

<u>Definition</u>. Given a natural number N and a set E, a function F will be said to be $B(N)$ on E if there is a number $M < \infty$ such that for any sequence $I_1,\dots,I_k,\dots$ of nonoverlapping intervals with $E \cap I_k \neq \emptyset$ there exist intervals J_{kn}, $n = 1,\dots,N$, such that

$$B(F;E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < M.$$

(Here $B(F;X)$ is the graph of F on the set X.)

We denote by $V_N(F;E)$ the infimum of the set of all numbers M appearing in the preceding definition.

<u>Lemma 1</u>. *Let $[a_i,b_i]$, $i = 1,2,\dots,$ be a sequence of nonoverlapping intervals, $b_i \le a_{i+1}$. Let $b = \sup b_i$ and let F be a function which is $B(N)$ on a set E with $E \cap [a_i,b_i] \neq \emptyset$. Then*

$$\sum_{i=1}^{\infty} V_N(F;E \cap [a_i,b_i]) \le V_N(F;E \cap [a_1,b]).$$

Proof. Let $V_i < V_N(F;E \cap [a_i, b_i])$. Then there is a sequence of nonoverlapping intervals

$$(I_k^i)_k, \qquad I_k^i \cap E \neq \emptyset, \qquad I_k^i \subset [a_i, b_i],$$

such that for any intervals

$$J_{kn}^i, \qquad n = 1, \ldots, N,$$

with

$$B(F;E \cap \bigcup_k I_k^i) \subset \bigcup_k \bigcup_{n=1}^N (I_k^i \times J_{kn}^i)$$

we have

$$\sum_k \sum_{n=1}^N |J_{kn}^i| \geq V_i. \tag{1}$$

Let $M > V_N(F;E \cap [a_1, b])$. There exist intervals

$$J_{kn}^i \ (k = 1,2,\ldots, \quad i = 1,2,\ldots, \quad n = 1,\ldots,N)$$

such that

$$B(F;E \cap \bigcup_k \bigcup_i I_k^i) \subset \bigcup_k \bigcup_i \bigcup_{n=1}^N (I_k^i \times J_{kn}^i)$$

and

$$\sum_k \sum_i \sum_{n=1}^N |J_{kn}^i| < M.$$

By (1) we have $\sum V_i < M$ which easily implies our assertion.

Let C denote the Cantor ternary set, i.e., $C = \{x : x = \sum c_i/3^i$ with c_i taking the values 0 and 2 only$\}$. Each point $x \in C$ is uniquely represented by $\sum c_i(x)/3^i$. Let Φ, F_1 and F_2 be functions defined as follows: for each

$x \in C$, $\Phi(x) = \Sigma\, c_i(x)/2^{i+1}$, $F_1(x) = \Sigma\, c_{2i-1}(x)/4^i$ and $F_2(x) = (1/2)\,\Sigma\, c_{2i}(x)/4^i$. Extending Φ, F_1 and F_2 linearly on each interval contiguous to C, one has Φ, F_1 and F_2 defined and continuous on $[0,1]$ (cf. [1]); Φ is the Cantor ternary function.

Remark 1. By [1] we have

$$F_1(x) = \begin{cases} (1/2)F_2(3x), & \text{if } x \in [0, 1/3] \\ x - (1/6), & \text{if } x \in (1/3, 2/3) \\ (1/2) + (1/2)F_2(3x-2), & \text{if } x \in [2/3, 1] \end{cases}$$

Lemma 2. $V_2(F_2;C) = 1$ and $V_2(F_1;C) = 1$.

Proof. In [1] V. Ene has shown that for any interval $[a,b]$, $a,b \in C$, there exist two intervals J_1 and J_2 such that

(2) $$B(F_2;C \cap [a,b]) \subset [a,b] \times (J_1 \cup J_2) \qquad \text{and}$$

$$|J_1| + |J_2| \le \Phi(b) - \Phi(a).$$

By (2) it follows that

(3) $$V_2(F_2;C) \le \Phi(1) - \Phi(0) = 1.$$

Let $[a_i,b_i]$, $i = 1,\dots,16$, be the closed intervals remaining after the 4th step in Cantor's ternary process. Let $V = V_2(F_2;C)$. Since $F_2(x) = (1/16)F_2(3^4(x-a_i)) + F_2(a_i)$ for each $x \in [a_i,b_i]$, it follows that

(4) $$V_2(F_2;[a_i,b_i] \cap C) = (1/16)V.$$

Now consider the closed interval $[b_7,a_{10}]$ for which we have $F_2([b_7,a_{10}] \cap C) \subset J_1 \cup J_2$ with $J_1 = [F_2(b_7),F_2(b_8)]$ and $J_2 = [F_2(a_9),F_2(a_{10})]$. We have

(5) $|J_2| + |J_2| = 2(1/16)$

and this sum is minimum. Applying Lemma 1 for the intervals $[b_7, a_{10}]$, $[a_i, b_i]$, $i \in \{1,2,\ldots,7\} \cup \{10,11,\ldots,16\}$, by (4) and (5), $(14/16)V + (2/16) \leq V$. Hence $V \geqq 1$ so that, by (3), $V = 1$. Since (2) is also true for F_1, by Remark 1 and Lemma 1 it follows now that $V_2(F_1;C) = 1$.

Remark 2. Let

$$I_{k_1,\ldots,k_n}, \qquad k_i = 1,2,3,4, \qquad i = 1,2,\ldots,n,$$

be the closed intervals remaining after the 2n-th step in Cantor's ternary process (numbered from left to right), and let

$$I_{k_1,\ldots,k_n} = [a_{k_1,\ldots,k_n}, b_{k_1,\ldots,k_n}].$$

Then for each $x \in C$,

$$x = I_{k_1(x)} \cap I_{k_1(x),k_2(x)} \cap \ldots \qquad \text{and}$$

$$F_2(x) = (1/4^n)F_2(9^n(x-a_{k_1,\ldots,k_n})) + F_2(a_{k_1,\ldots,k_n}).$$

Theorem 1. F_1 and F_2 are B(2) on C and for each $x \in C$ we have : $V_2(F_2;[0,x] \cap C) = \Phi(x)$ and $V_2(F_1;[0,x] \cap C) = \Phi(x)$.

Proof. By [1] it follows that F_1 and F_2 are B(2) on C. By (2), $V_2(F_2;[0,x] \cap C) \leq \Phi(x)$. By Remark 2 and Lemma 2,

$$V_2(F_2;I_{k_1,\ldots,k_n} \cap C) = (1/4^n).$$

Now by Lemma 1, $V_2(F_2;[0,x] \cap C) \geq k_1(x)/4 + k_2(x)/4^2 + \ldots = \Phi(x)$. By (2), Remark 1 and Lemma 1 it follows that $V_2(F_1;[0,x] \cap C) = \phi(x)$.

Remark 3. Let n_o be a natural number, $n_o \geq 2$. Then for each $i = 0,1,\ldots,n_o-1$ we define

$$G_i(x) = \sum_{k=1}^{\infty} c_{kn_o+i}(x)/2^{kn_o+i+1}, \qquad x \in C.$$

By an argument analogous to the proof of Theorem 1 one can show that G_i is $B(2)$ on C and $V_2(G_i;[0,x] \cap C) = \phi(x)$ for each $x \in C$.

Theorem 2. There exist continuous functions satisfying Lusin's condition (N) on $[0,1]$ which are $B(N)$ on C for no natural number N.

Proof. Let $q \in (2,4)$ and let F_q be defined as follows: for each $x \in C$, $F_q(x) = \Sigma\, c_{2k}(x)/q^k$, and extending F_q linearly on each interval contiguous to C one has F_q defined and continuous on $[0,1]$. $F_q(C)$ can be covered by 2^n closed intervals, each of length at most $2/q^n$. Hence $|F_q(C)| \leq 2^n(2/q^n)$ so that $|F_q(C)| = 0$. Let $[a_i,b_i]$, $i = 1,2,3,4$, be the closed intervals remaining after the second step of Cantor's ternary process. Suppose that F_q is $B(N)$ on C for some N and let $V = V_N(F_q;C)$. We have $F_q(x) = (1/q)F_q(9(x-a_i)) + F_q(a_i)$ for each $x \in [a_i,b_i]$. Then $V_i = V_N(F_q;C \cap [a_i,b_i]) = V/q$. By Lemma 1, $\Sigma\, V_i \leq V$. Hence $4(V/q) \leq V$ and $q \geq 4$, which is a contradiction.

We are indebted to Professor Solomon Marcus for the help given in preparing this article.

References

[1] V. Ene: On Foran's conditions A(N), B(N) and (M).
Real Analysis Exchange, Vol. 9, No. 2(1984), 495-501.

[2] J. Foran: An extension of the Denjoy integral.
Proc. Amer. Math. Soc., 49(1975), 82-91.

Received March 26, 1984