P. Isaza, Department of Mathematics, Syracuse University, Syracuse, New York 13210

A SPACE OF REGULATED FUNCTIONS WHOSE FOURIER SERIES ARE EVERYWHERE CONVERGENT

1. Let f be a regulated function (i.e. $f(x) = \frac{1}{2}(f(x+0)+f(x-0))$) on the unit circle $T = [0,2\pi)$, and $I_1,\ldots,I_{2n}$ a cyclically ordered collection of contiguous intervals of length π/n, forming a partition of T. If $I_i = [a_i,b_i]$ we write $f(I_i) = f(b_i)-f(a_i)$. Let $V_n(f)$ be the supremum of the sums $\Sigma|f(I_i)|/i$ over such collections $\{I_i\}$, and let $V(f) = \sup_n V_n(f)$.

Let $x \in T$, $\delta > 0$. For a given n and $0 \le t < \pi/n$, let $I_i^+(t) = [x+t+\pi/n,\ x+t+(i+1)\pi/n]$ and $I_i^-(t) = [x-t-(i+1)\pi/n, x-t-i\pi/n]$, $i = 1,2,\ldots,N$, where $N = [n\delta/\pi]$. Define

$$\begin{aligned} V_n(f,x,\delta) &= V_n(x,\delta) \\ &= \sup\Big\{\max\Big[\sum_{i=1}^{N-j} |f(I_{j+i}^+(t))|/i,\ \sum_{i=1}^{j+1} |f(I_{j+2-i}^+(t))|/i, \\ &\qquad \sum_{i=1}^{N-j} |f(I_{j+i}^-(t))|/i,\ \sum_{i=1}^{j+1} |f(I_{j+2-i}^-(t))|/i\Big] : \\ &\qquad 0 \le j < N,\ 0 \le t < \pi/n\Big\} \end{aligned}$$

and $V(f,x,\delta) = V(x,\delta) = \limsup_n V_n(f,x,\delta)$.

We will consider the space of functions f for which $V(f,x,\delta) \to 0$ as $\delta \to 0$ for all x. Clearly, if f satisfies this condition, then $V(f) < \infty$, and it is easily seen that this space is a Banach space under the norm $\|f\| = \sup_x |f(x)| + V(f)$.

2. A function f is said to be of ordered harmonic bounded variation (OHBV) on $[a,b]$, if there is an M such that the sums $\Sigma |f(I_i)|/i < M$ for all finite collections $\{I_i\}$ of nonoverlapping intervals $I_i \subset [a,b]$ ordered from left to right or from right to left. Let $V(f,[a,b])$ be the supremum of such sums. For $(a,b]$ let $V(f,(a,b]) = \lim_{y \to a^+} V(f,[y,b])$. Similarly $V(f,[a,b)) = \lim_{y \to b^-} V(f,[a,y])$.

We will show that the space $Y = \{f \in \text{OHBV} : V(f,(x,x+\delta]) \to 0$ and $V(f,[x-\delta,x)) \to 0$ as $\delta \to 0$ for all $x\}$ is properly contained in the space $X = \{f | V(x,\delta) \to 0$ as $\delta \to 0$ for all $x\}$. It is clear that $Y \subseteq X$.

3. Definition. Let $[a,b] \subset (c,d) \subset (0,2\pi)$, $d-b$, $a-c > b-a$, and let φ be a function defined on (c,d) such that for some K, $|\varphi(I)| \leq K|I|$ for all intervals $I \subset (c,d)$, and $\varphi(x) = 0$ if $x \in (c,d)-(a,b)$. Let Q be a partition of T into intervals of length $r < b-a$, and $I_1, \ldots, I_M$ those intervals of Q which intersect $[a,b]$ ordered from left to right. Let $A_r(\varphi,[a,b])$ be the supremum of the sums

$$\sum_{i=1}^{M-j} |\varphi(I_{j+i})|/i \quad \text{and} \quad \sum_{i=1}^{j+1} |\varphi(I_{j+2-i})|/i$$

over all $0 \leq j < M$ and all partitions Q. Let $A(\varphi,[a,b]) = \sup_r Ar(\varphi,[a,b])$. Ar is defined for all r and $Ar \to 0$ as $r \to 0$; $Ar(w\varphi,[a,b]) = |w| Ar(\varphi,[a,b])$ and if $\psi(x) = \nu x+v$, $\nu \neq 0$, then $Ar(\varphi,[a,b]) = A(r/\nu)(\varphi \circ \psi, [(a-v)/\nu, (b-v)/\nu])$.

4. We now construct a function $f \in X$ which does not belong to Y. Let $\varphi(x)=0$ if $x \in [-1,0] \cup [1,2]$, $\varphi(x)=2x$ if $x \in [0,1/2]$ and $\varphi(x)=-2(x-1)$ if $x \in [1/2,1]$. Choose $m_1 > 2$ and define $f(x)=0$ if $x \in [\pi/2m_1, 3\pi/4m_1] \cup [\pi/m_1, 2\pi)$, and $f(x)=\varphi(4m_1(x-3\pi/4m_1)/\pi)/\log 2$ if $x \in (3\pi/4m_1, \pi/m_1)$. There is M_1 such that $|f(I)| < M_1|I|$ for $I \subset (\pi/2m_1, 2\pi)$.

If $m_2, \ldots, m_{k-1}$ have been chosen such that f is defined in $[\pi/2m_{k-1}, 2\pi)$ and there is M_{k-1} satisfying $|f(I)| < M_{k-1}|I|$ for $I \subset (\pi/2m_{k-1}, 2\pi)$, then we can choose $m_k > 4m_{k-1}$ such that $Ar(f,[3\pi/4m_{k-1}, \pi/2]) < 1/k$ for $r < \pi/m_k$, and define f in $[\pi/2m_k, \pi/2m_{k-1}]$ by $f(x)=0$ for $x \in [\pi/2m_k, 3\pi/4m_k] \cup [\pi/m_k, \pi/m_{k-1}]$, and $f(x)=\varphi(4m_k(x-3\pi/4m_k)/\pi)/\log(k+1)$ if $x \in [3\pi/4m_k, \pi/4m_k]$. Then there is M_k such that $|f(I)| < M_k|I|$ for $(\pi/2m_k, 2\pi)$. Setting $f(0)=0$, f is defined in $[0,2\pi)$.

To prove that $V(f,x,\delta) \to 0$ as $\delta \to 0$ for all x, we first notice that f is of bounded variation in a neighborhood of each $x \neq 0$, and therefore it is enough to show that $V(0,\delta) \to 0$ as $\delta \to 0$.

Let $\varepsilon > 0$ and s so large that $(4+A(\varphi[0,1]))/\log s + 1/(s-1) < \varepsilon$. For $\delta < \pi/m_{s-1}$, choose n such that $\pi/n < \delta$, and let $I_1^+(t), \ldots, I_N^+(t)$ be as in §1. Let k be the smallest i such that $\pi/m_i \leq \pi/n$, then $\pi/n < \pi/m_{k-1}$. Only $I_1^+(0)$ can intersect $[0, \pi/m_k]$; and if it does, since $m_k > 4m_{k-1}$, $I_1^+(0) \cap [3\pi/4m_{k-1}, \pi/m_{k-1}] = \emptyset$. Let $L_1, \ldots, L_t$ be the intervals $I_i^+(t)$ intersecting $[3\pi/4m_{k-1}, \pi/m_{k-1}]$. If $\pi/n \geq \pi/4m_{k-1}$, then $t \leq 3$. If $\pi/n < \pi/4m_{k-1}$, then all the L_i's are contained in

$(\pi/2m_{k-1}, 5\pi/4m_{k-1})$. Also we have that $L_i \cap [3\pi/4m_{k-2}, \delta) = \emptyset$. Finally, let $M_1, \ldots, M_\ell$ be the intervals $I_i^+(t)$ intersecting $[3\pi/4m_{k-2}, \delta)$. Thus we have the estimate

$$\sum_{i=1}^{N-j} |f(I_{j+i}^+(t))|/i \le 1/\log(k+1) + \sum_{i=1}^{t} |f(L_i)|/i + \sum_{i=1}^{\ell} |f(M_i)|/i$$

$$\le 1/\log(k+1) + 3/\log k + A(f, [3\pi/4m_{k-1}, \pi/m_{k-1}]) + 1/(k-1)$$

$$\le (4+A(\varphi, [0,1]))/\log s + 1/(s-1) < \epsilon .$$

The same estimate holds for $\sum_{i=1}^{j+1} |f(I_{j+2-i}^+(t))|/i$. Hence $V(0,\delta) \to 0$ as $\delta \to 0$. However $f \notin$ OHBV.

5. We have mentioned that if $V(f,x,\delta) \to 0$ as $\delta \to 0$ for all x, then $V(f) < \infty$. However, the converse is not true.

Consider the sequence $\varphi_1, \varphi_2, \ldots$ defined by $\varphi_1(x) = \varphi(x)$, $\varphi(x)$ as in §4 and $\varphi_i(x) = 0$ if $x \in [-1,0] \cup [1,2]$, and $\varphi_i(x) = \varphi(ix-j)$ if $j/i \le x < (j+1)/i$, $j = 0,1,\ldots,(i-1)$. Let $A(\varphi_1, [0,1]) = A$. We can easily see that

$$\sum_{i=1}^{2k} 1/i \le A(\varphi_k) \le A(\varphi_{k-1}) + A, \qquad k = 2,3,\ldots .$$

Then there is a subsequence $\{\varphi_{k_j}\}$ such that $jA \le A(\varphi_{k_j}, [0,1]) \le (j+1)A$. Let $\psi_i = \varphi_{k_i}$. For each k there is a rational $r_k = p_k/q_k < 1$ and contiguous intervals $I_1, \ldots, I_m$ of length r_k in $(-1,2)$ such that

$$(*) \qquad \sum_{i=1}^{M} |\psi_k(I_i)|/i > A(\psi_k, [0,1])/2 .$$

We define now f following the inductive procedure of §4, choosing m_k as before, but also a multiple of p_k, say $m_k = p_k \ell_k$ and defining $f(x) = \psi_k(4m_k(x-3\pi/4m_k)/\pi)/k$ if $x \in [3\pi/4m_k, \pi/m_k]$. We can see as in §4 that $V(f) < \infty$. However, for $\delta > 0$, if we let s be so large that $\pi/m_s < \delta/2$, by (*) there are continguous intervals $J_1, \ldots, J_m$ of length $(\pi/4m_s)(p_s/q_s) = \pi/4q_s\ell_s$, in $(\pi/2m_s, 5\pi/4m_s) \subset (0,\delta)$, such that

$$\sum_{i=1}^{M} |f(J_i)|/i = \frac{1}{s} \sum_{i=1}^{M} |\psi_s(I_i)|/i > A(\psi_s)/2s \geq sA/2s = A/2 .$$

Therefore for $n = 4q_s\ell_s$, an appropriate value of $0 \leq t < \pi/n$ and some j we will have that

$$\sum_{i=1}^{N-j} |f(I^{+}_{j+i}(t))|/i \geq \sum_{i=1}^{M} |f(J_i)|/i > A/2 ,$$

and thus $V_{4q_s\ell_s}(0,\delta) > A/2$. Since we can take s, and therefore m_s, arbitrarily large, and $n = 4q_s\ell_s \geq 4p_s\ell_s = 4m_s$, it follows that for arbitrarily large values of n, $V_n(0,\delta) > A/2$. Hence $\limsup_n V_n(0,\delta) \geq A/2$ for all δ.

6. Theorem. If $V(f,x,\delta) \to 0$ as $\delta \to 0$ for all x, then the $n^{\underline{th}}$ partial sum of the Fourier series of f at x, $s_n(f,x) \to f(x)$ for all x. If f is continuous on a closed interval I, the convergence is uniform on each interval J contained in the interior of I.

Proof. For $h(t) = h(x,t) = f(x+t) + f(x-t) - 2f(x)$ and $\delta > 0$,

$$s_n(f,x)-f(x) = \frac{1}{\pi}\int_0^{\delta} h(t)\,\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{1}{\pi}\int_0^{\pi/n} h(t)\,\frac{\sin nt}{t}\,dt + \frac{1}{\pi}\int_0^{\pi/n} \sum_{i=1}^{(N-1)/2} \left[\frac{h(t+2i\pi/n)}{t+2i\pi/n} - \frac{h(t+(2i-1)\pi/n)}{t+(2i-1)\pi/n}\right]$$

$$\sin nt\,dt + \int_N^{\delta} h(t)\,\frac{\sin nt}{t}\,dt = I + II + III\,,$$ where N is the largest odd integer less than $n\delta/\pi$.

I and III are easily seen to be $o(1)$.

Now

$$|II| \le \frac{1}{\pi}\int_0^{\pi/n} \sum_{i=1}^{(N-1)/2} \frac{|h(t+2i\pi/n)-h(t+(2i-1)\pi/n)|}{t+2i\pi/n}\,dt$$

$$+ \frac{1}{\pi}\int_0^{\pi/n} \sum_{i=1}^{(N-1)/2} \frac{|h(t+(2i-1)\pi/n)|}{\frac{\pi}{n}\,i^2}\,dt$$

$$\le \frac{2}{\pi}V_n(f,x,\delta) + \frac{1}{\pi} \sup_{\frac{\pi}{n}\le t\le 2[\sqrt{n}]\pi/n} |h(t)| \sum_{i=1}^{\infty} 1/i^2$$

$$+ \frac{1}{\pi}\sup_{t\in T}|h(t)| \sum_{i=[\sqrt{n}]}^{\infty} 1/i^2$$

$$= \frac{2}{\pi}V_n(f,x,\delta) + o(1)\,.$$

Thus $|s_n(f,x)-f(x)| \le \frac{2}{\pi}V_n(f,x,\delta) + o(1)$ and so $\overline{\lim}|s_n(x)-f(x)| \le \frac{2}{\pi}V(x,\delta)$. Since δ can be taken arbitrarily small, we have that $\overline{\lim}|s_n(x)-f(x)| = 0$.

If f is continuous in a closed interval I, the above estimates are $o(1)$ uniformly in $x \in J \subset \operatorname{int} I$. Also, by an argument of compactness we can see that $V(x,\delta) \to 0$, uniformly in x, as $\delta \to 0$.

REFERENCES

1. D. Waterman, Λ-bounded variation: recent results and unsolved problems, Real Analysis Exchange 4(1978-1979), 69-75.

2. D. Waterman, Fourier series of functions of Λ-bounded variation, Proc. Amer. Math. Soc. 74(1979), 119-123.