The packing measure of rectifiable sets†

by

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In the theory of Lebesque measure in $\mathbb{R}^d$, the density theorem plays an important role in connecting the abstract Radon-Nikodym theorem with the local definition of a limit. There is a natural fit between Lebesque measure in $\mathbb{R}^k$ and the local geometry of a measurable set. But there are other measures in $\mathbb{R}^k$, based on outer measures which charge sets of zero Lebesque measure. Hausdorff exploited the Carathéodory method of construction to define a translation invariant measure in $\mathbb{R}^k$, based on coverings which are economical. Let us recall one version of his definition: let $\phi:(0,\delta) \to \mathbb{R}^+$ be monotone increasing with $\phi(0+) = 0$;

$$(1) \qquad \phi - m(E) = \lim_{\delta \downarrow 0} \inf\{\Sigma\phi(2r_i) : E \subset \bigcup_{i=1}^{\infty} B(x_i,r_i),\ \ r_i < \delta\}$$

where $B(x_i,r_i)$ denotes the open ball centre x_i, radius $r_i > 0$, defines a metric outer measure on the power set of $\mathbb{R}^k$. Restricting this to the class $\mathcal{M}_\phi$ of measurable subsets gives (spherical) Hausdorff ϕ-measure. In particular, if $\phi(s) = cs^\alpha$, you get s-dimensional measure in $\mathbb{R}^d$.

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†This is a survey of recent results obtained in collaboration between the two authors, and summarized in a lecture at the Symposium given by the first author.

For $\alpha < d$, $\mathbb{R}^d$ has non σ- finite s^α-measure, but we can look for a density theorem valid for subsets of positive finite measure. For example if E is the classical Cantor set in [0,1] and $\alpha = \log 2/\log 3$, Hausdorff proved that

$$s^\alpha\text{-}m(E) = 1.$$

However direct calcuation shows that, for $x \in E$, $0 \leq \liminf_{\substack{x \in I \\ |I| \to 0}} \frac{s^\alpha\text{-}m(E \cap I)}{|I|^\alpha} <$ $1 = \limsup_{\substack{x \in I \\ |I| \to 0}} \frac{s^\alpha\text{-}m(E \cap I)}{|I|^\alpha}$, so that no strong analogue of the Lebesgue density theorem holds even for an extremely regular set like the Cantor set. In fact Besicovitch [1] shows that for subsets of $\mathbb{R}^1$, $0 < \alpha < 1$, there is no set E with $0 < s^\alpha\text{-}m(E) < \infty$ for which $\lim_{\substack{x \in I \\ |I| \to 0}} \frac{s^\alpha\text{-}m(E \cap I)}{|I|^\alpha}$ exists a.e., and Marstrand [5] shows that a density can only exist a.e. for a subset of $\mathbb{R}^2$ if $\alpha = 0,1$, or 2.

The corresponding result in $\mathbb{R}^k$ must be true, but I am not aware that it has been proved rigorously-namely, that it is only for integer powers that one can hope to have a density for <u>any</u> set. When α is an integer $0 < \alpha < k$, the sets E for which $0 < s^\alpha\text{-}m(E) < +\infty$ and the density holds a.e. in E are in some sense locally 'almost embedded' in a suitable ℓ-dimensional subspace. They were first studied by Besicovitch [1] in the case $\alpha = 1$, $k = 2$ and called regular. These Besicovitch results were extended by many authors to general (α,k) and are described in the book by Federer [4] on geometric measure theory. The reader who would like a more elementary organized account is advised to look in the forthcoming book by Falconer [3].

Let me summarize some of the Besicovitch results for linear measure (Hausdorff measure with $\phi(s) = s$) on $\mathbb{R}^2$, which we now denote by m. Suppose E is m-measurable, $0 < m(E) < +\infty$; then E is regular if $\frac{m(E \cap B(x,r))}{2r} \to 1$ for m a.e. $x \in E$. E is (purely) irregular if $\frac{m(E \cap B(x,r))}{2r} \not\to 1$ for m a.e. $x \in E$ and any E can be expressed as the union of a regular E_1 and an irregular E_2.

Further it is proved that E is regular if and only if one of the following holds, apart from a set of zero m-measure:

(a) E is a countable union of subsets of rectifiable arcs

(b) E has a tangent a.e.

(c) in all directions except at most 1, E projects onto a set of positive Lebesque (linear) measure.

On the other hand E is irregular if one of the following holds:

(a) no rectifiable arc intersects E in a set of positive measure

(b) set of points where E has a tangent has zero measure

(c) in almost all directions, E projects onto a set of zero Lebesque (linear) measure.

Geometric measure theory is an elaborate and useful development of techniques based on these results. However there is at least one important sense in which the Besicovitch regular sets in $\mathbb{R}^2$ are <u>not</u> 'curvelike'. If C, D are two rectifiable arcs in $\mathbb{R}^2$, then the Cartesian product $C \times D$ is a rectifiable 2-surface in $\mathbb{R}^4$. However, if A, B are regular linearly measurable sets in $\mathbb{R}^2$, it does <u>not</u> follow that $A \times B$ is a 2-regular subset of $\mathbb{R}^4$, in fact it may not even have Hausdorff dimension 2: examples can be given of regular sets A, B with $\dim A \times B = 3$. The reason for this difficulty is that Besicovitch allows an exceptional set of zero m-measure in his definition of regularity. These exceptional sets can cause difficulities in any context for which m-measure is not the appropriate definition of smallness. In order to control the exceptional sets we need a new definition of measure which, in general, gives a larger value than Hausdorff measure.

Let us summarize the definition of such a measure, which we call packing measure. Put

$$(2) \qquad \phi - P(E) = \lim_{\delta \downarrow 0} \sup\{\Sigma\phi(2r_i) : B(x_i, r_i) \text{ disjoint}, x_i \in E, r_i < \delta\}$$

This defines a set function which is monotone, but not countably subadditive, since ϕ-P$\{x_0\}$= 0 but $s^{\frac{1}{2}}$-P(D) = $+\infty$ for the countable compact set $D = \{0, k^{-1}; k \in \mathbb{N}\}$. It is not an outer measure but it is a pre-measure so we can use method I of Munroe to give

$$(3) \qquad \begin{aligned} \phi - p(E) &= \inf\{\Sigma\phi\text{-}P(E_i) : E \subset \bigcup E_i\} \\ &= \inf\{\lim \phi\text{-}P(E_n) : E_n \uparrow E\}. \end{aligned}$$

The measure theory for ϕ-p is studied in [7]. For all $E \subset \mathbb{R}$, ϕ-m(E) $\leq$ ϕ-p(E). For the Cantor set $E \subset \mathbb{R}$ with $\phi(s) = s^\alpha$, $\alpha = \log 2/\log 3$

$$s^\alpha\text{-m}(E) = 1 \quad \text{but} \quad s^\alpha\text{-p}(E) = 2.$$

This leads us to conjecture that, if there is a set $E \subset \mathbb{R}^d$ with $0 < s^\alpha\text{-m}(E) = s^\alpha\text{-p}(E) < +\infty$ then α is an integer, and E is regular.

For any set $E \subset \mathbb{R}^k$, we have the Hausdorff dimension

$$\dim E = \inf\{\alpha>0: s^\alpha\text{-}m(E)=0\},$$

and we can now define the packing dimension

$$\mathrm{Dim}\ E - \inf\{\alpha>0: s^\alpha\text{-}p(E)=0\}.$$

Clearly,

$$0 \leq \dim E \leq \mathrm{Dim}\ E \leq k,$$

and for any α,β satisfying $0 \leq \alpha \leq \beta \leq k$, it is not difficult to construct a Cantor like set $E \subset \mathbb{R}^k$ with $\dim E = \alpha$, $\mathrm{Dim}\ E = \beta$. For a Lebesque measurable set $E \subset \mathbb{R}^k$ of finite positive Lebesque measure, it is not difficult to prove that $s^k - m(E) = s^k - p(E) = c|E|$.

We can now consider three distinct notions of regularity for a subset $E \subset \mathbb{R}^k$.

(a) E is <u>weakly regular</u> if $\dim E = \mathrm{Dim}\ E$.

We remark that this is useful when we consider Cartesian products. In general, see Eggleston [2],

$$\dim A \times B \leq \dim A + \dim B$$

but if at least one of A,B is weakly regular, see Tricot [9], then

$$\dim A \times B = \dim A + \dim B.$$

(b) E is <u>ϕ-regular</u> if $0 < \phi - m(E) < +\infty$, and the density theorem holds $\phi - m$ a.e. There are many equivalent definitions, all allowing for an exceptional set of zero ϕ-measure, in the sense of Hausdorff.

(c) E is <u>strongly ϕ-regular</u> if

$$(4) \qquad 0 < \phi - m(E) = \phi - p(E) < +\infty.$$

If our conjecture is valid, this forces $\phi(s) = s^r$ with r an integer. We justify the definition by exploring its implications for $\phi(s) = s$ and subsets of the plane. To simplify notation $s - p(E)$ will be denoted $p(E)$. The remaining results are proved in detail in [8], so we just summarize them here. It is convenient to define upper and lower (linear) densities for an arbitrary measure μ, defined on Borel sets in $\mathbb{R}^2$.

$$(5) \qquad \begin{aligned} \underline{D}_\mu(x) &= \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} \\ \bar{D}_\mu(x) &= \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} \end{aligned}$$

Morse and Randolph [6] studied these functions using (Hausdorff linear) measure m; we now restate their results, and obtain corresponding ones for the (linear packing) measure p.

$$(6) \qquad p(E) < +\infty \Rightarrow \mu(E) \geq p(E) \inf\{\underline{D}_\mu(x) : x \in E\}$$

$$(7) \qquad \mu(E) \leq p(E) \sup\{\underline{D}_\mu(x) : x \in E\}$$

$$(8) \qquad \mu(E) \geq m(E) \inf\{\bar{D}_\mu(x) : x \in E\}$$

(9) $$\mu(E) \leq 2m(E) \sup\{\bar{D}_\mu(x) : x \in E\}$$

We remark that the factor 2 in (9) is required because there is a set E_0 with $0 < m(E_0) < \infty$, but $\bar{D}_\mu(x) = \frac{1}{2}$ for all $x \in E_0$ with $\mu(E) = m(E \cap E_0)$. This factor arises because in the definition of m, we do not cover by balls centred in E, while the computation of density only uses centred balls.

We can extend our definitions for density of E at x, and obtain them as special cases of (6) to (9) by taking μ to be the restriction of p-measure or m-measure to E. Thus

$$\bar{D}(x,E) = \lim\sup \frac{m(E \cap B(x,r))}{2r}$$

$$\underline{D}(x,E) = \lim\inf \frac{m(E \cap B(x,r))}{2r}$$

$$\bar{\Delta}(x,E) = \lim\sup \frac{p(E \cap B(x,r))}{2r}$$

$$\underline{\Delta}(x,E) = \lim\inf \frac{p(E \cap B(x,r))}{2r}$$

and we get immediate corollaries of (6) - (9). We now distinguish between exceptional sets by writing m.a.s. (resp p.a.s.) when the statement is true outside a set of m-measure (resp p-measure) zero. We state these formally

Corollary 1. If E is m-measurable, $0 < m(E) < +\infty$, then $\frac{1}{2} \leq \bar{D}(x,E) \leq 1$ m.a.s.

Corollary 2. If E is p-measurable, $0 < p(E) < +\infty$, then $\underline{\Lambda}(x,E) = 1$ p.a.s.

Corollary 3. If E is p-measurable, $0 < p(E) < +\infty$, then $m(E) = 0 \Longleftrightarrow$ $\bar{\Delta}(x,E) = +\infty$ p.a.s.

Corollary 4. If E, F are m-measurable, $E \subset F$ and $0 < m(F) \leq m(E) < +\infty$, then m.a.s. on F we have $\underline{D}(x,F) = \underline{D}(x,E)$ and $\bar{D}(x,F) = \bar{D}(x,E)$.

If, in addition $p(E) < \infty$, then m.a.s. on F, $\underline{\Delta}(x,F) = \underline{\Delta}(x,E)$ and $\bar{\Delta}(x,F) = \bar{\Delta}(x,E)$.

Corollary 5. If E is p-measurable, $0 < p(E) < \infty$ and $\bar{\Delta}(x,E) < +\infty$ p.a.s., then $F \subset E$, $m(F) = 0 \Rightarrow p(F) = 0$.

We can now work out the implications of the condition of strong regularity $0 < m(E) = p(E) < +\infty$.

Theorem 1. If E is arcwise connected and $m(E) < +\infty$, then E is strongly regular.

Theorem 2. A set E with $0 < m(E) < +\infty$ is strongly regular if and only if one of the following holds

(a) $\underline{D}(x,E) = 1$ p.a.s. on E;

(b) $\bar{\Delta}(x,E) = 1$ p.a.s. on E;

(c) $E = E_1 \cup E_2$ where E_1 is a subset of a countable union of rectifiable arcs and $p(E_2) = 0$;

(d) E has a tangent at x for p.a.s. x in E.

It is easy to give examples of sets E which are regular but not strongly regular, but if $0 < p(E) < \infty$ and $\bar{\Delta}(x,E) < \infty$ p.a.s., then regularity and strong regularity are equivalent. The slight strengthening in the condition gives control over the exceptional set in the original Besicovitch definition. Our hope is that it will also lead to some simplifications in geometric measure theory.

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