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Products of Blackwell spaces and regular conditional probabilities

§0. Introduction

The study of Blackwell properties has attracted a number of researchers, a list of which may be found in the monograph of B. V. Rao and K. P. S. Bhaskara Rao [1] . More recent efforts are due to W. Bzyl [3], [4], J. Jasiński [7], [8], D. Fremlin [5], [6], and the author [14], [17], [18] . Most of the current work seeks to characterise the Blackwell property for specific classes of singular spaces occurring in abstract analysis and the descriptive theory of sets. The present excursion, taking up where [17] left off, explores the combinatorial behaviour of Blackwell spaces with totally imperfect complement. In particular, it answers the question of when a product of such spaces inherits the Blackwell property.

The results obtained supply a solution to a problem in probabilistic measure theory posed by D. Ramachandran: do regular conditional probabilities always exist over strong Blackwell domains? A negative answer is provided by corollary 3 *infra*.

§1. Preliminaries

Throughout this paper, the symbol S will denote an uncountable standard measurable space. One may view S as a Polish space and $\mathcal{B}(S)$ as the (Borel) σ-algebra generated by the open subsets of S. A σ-ideal $\mathcal{I}$ in $\mathcal{B}(S)$ is continuous if it contains all singleton subsets of S. We insist that S not be a member of $\mathcal{I}$. A subset X of S is $\mathcal{I}$-Lusin if X is uncountable and the intersection of X with every member of $\mathcal{I}$ is countable. (If $\mathcal{I}$ is the collection of all first category subsets of $S = \mathbb{R}$, then these are the classical Lusin sets [2].)

Let S^n denote the n-fold product of S with itself. If $A_1, \ldots, A_n$ are subsets of S, then $\langle A_1, \ldots, A_n\rangle$ denotes the set of all $(s_1, \ldots, s_n)$ in S^n such that $s_i \in A_i$ for some $i = 1, \ldots, n$. Thus $\langle A_1, \ldots, A_n\rangle$ is the complement of the rectangle $(S \setminus A_1) \times (S \setminus A_2) \times \ldots \times (S \setminus A_n)$.

A subset R of S^n is $\mathcal{I}$-reticulate if there are sets $N_1, \ldots, N_n$ in $\mathcal{I}$ such that $R \subset \langle N_1, \ldots, N_n\rangle$. A set T in $\mathcal{B}(S^n)$ is an $\mathcal{I}$-thread if

1) $\langle \phi, \ldots, \phi, A_i, \phi, \ldots, \phi\rangle \cap T$ is at most a singleton set whenever A_i is singleton (all i);

2) T is not $\mathcal{I}$-reticulate in $S \times S$.

A subset X of S is $\mathcal{I}$-dense (of order 1) if every $B \in \mathcal{B}(S)$ disjoint from X is a member of $\mathcal{I}$. Say that X is $\mathcal{I}$-dense of order n if X^n meets every R in $\mathcal{B}(S^n)$ which is not $\mathcal{I}$-reticulate. The following is easily checked:

Lemma 1: If X is $\mathcal{I}$-dense of order $n + 1$ in S, then X is

I-dense of order n $(n \geq 1)$.

Two particular σ-ideals will be of interest to us. The first consists of all countable subsets of S . This σ-ideal I is continuous, and its I-Lusin sets are precisely the uncountable subsets of S . For this σ-ideal, we use the following terminology:

General σ-ideal I	I = countable sets
I-Lusin sets	uncountable sets
I-reticulate set	reticulate set
I-dense	Borel-dense
I-thread	thread

This table connects the terminology used in this paper with that established in [11], [12], [15], [16], and [17] .

<u>Lemma 2</u>: Let R be an analytic subset of S^n which is not reticulate. Then R contains a thread.

<u>Indication</u>: This is proved by H. Sarbadhikari in [11] .

<u>Lemma 3</u>: The following are equivalent:

1) X is Borel-dense of order n in S ;

2) If A is an analytic subset of $S^n \setminus X^n$, then there are countable subsets $A_1, \dots, A_n$ of $S \setminus X$ with $A \subset \langle A_1, \dots, A_n \rangle$.

<u>Indication</u>: See Proposition 2 and the "coffin" on p. 188 in [12].

A subset N of S is said to have property (s°) if wherever B is an uncountable member of $\mathcal{B}(S)$, then there is another uncountable A in $\mathcal{B}(S)$ such that $A \subset B$ and $A \cap N = \emptyset$. This property was introduced by Marczewski in [9] and it is discussed by Brown and Cox in [2]. Proofs of the following lemmas are to be found in [9].

Lemma 4: Let S be a Polish space. If $N \subset S$ is either universally null or a set always of first category, then N has property (s°).

Lemma 5: A countable union of sets with property (s°) or a Borel-isomorph of a set with property (s°) again has property (s°).

With these facts in hand, it is possible to demonstrate the following:

Lemma 6: Suppose that X is a subset of S whose complement $S \setminus X$ has property (s°). Then X is Borel-dense of order n for every n.

Remark: This generalises Corollary 5 in [12].

Proof: Clearly, X is Borel-dense of order 1. If X is not Borel-dense of order n, then lemma 2 ensures the existence of a thread T of S^n contained in $S^n \setminus X^n$. Now T may be written as the union of the sets $[S^k \times (S \setminus X) \times S^{n-k-1}] \cap T$ for $k = 0, 1, \ldots, n-1$, each of which has property (s°): this follows from lemma 5 and the fact that each of these sets is an isomorph of some subset of $S \setminus X$. So it would result that T has property (s°), a contradiction.

Q.E.D.

Lemma 7: For each $n \geq 1$, there is a subset X of S which is Borel-dense of order n, but not of order $n+1$.

Indication: This is Proposition 3 in [12].

Besides the σ-ideal of countable sets, we shall also be concerned with another σ-ideal in $\mathcal{B}(S)$, defined as follows: Let X be a fixed uncountable subset of S. Define $\mathcal{I}(X)$ to be the σ-ideal consisting of all B in $\mathcal{B}(S)$ with $B \cap X$ countable. Then $\mathcal{I}(X)$ is continuous, and X is an $\mathcal{I}(X)$-Lusin set $\mathcal{I}(X)$-dense of order 1. Note that $\mathcal{I}(X)$ is the σ-ideal of all countable sets just in case X is Borel-dense.

A separable measurable space $(X, \mathcal{B}(X))$ has the Blackwell property if whenever $\mathcal{C}$ is a countably generated sub-σ-algebra of $\mathcal{B}(X)$ separating points of X, then necessarily $\mathcal{C} = \mathcal{B}(X)$. Equivalently, every one-one $\mathcal{B}(X)$-measurable function on X is a Borel-isomorphism on X. Say that $(X, \mathcal{B}(X))$ has the strong Blackwell property if whenever $\mathcal{C}$ and $\mathcal{D}$ are countably generated sub-σ-algebras of $\mathcal{B}(X)$ with the same atoms, then necessarily $\mathcal{C} = \mathcal{D}$. For equivalent definitions and a general discussion of these properties, see [1].

The fundamental tool used here is the next lemma, which follows from results in [18], viz. Propositions 1 and 2.

Lemma 8: Suppose that $\mathcal{I}$ is a continuous σ-ideal in the Borel structure $\mathcal{B}(S)$. Let X be an $\mathcal{I}$-Lusin set $\mathcal{I}$-dense in S. Then each of the following conditions implies its successor:

1) X is $\mathcal{I}$-dense of order 2 in S;

2) X is strongly Blackwell;

3) X is Blackwell;

4) $X \times X$ intersects every $\mathcal{I}$-thread in $S \times S$.

If I is the σ-ideal of countable sets, then all four conditions are equivalent.

§2. Main Results

Lemma 9: Let n be a positive integer. Then each of the following conditions implies its successor:

1) X is Borel-dense of order $2n$ in S ;

2) X^n is $I(X^n)$-dense of order 2 in S^n ;

3) $X^n \times X^n$ meets each $I(X^n)$-thread in $S^n \times S^n$.

If X is Borel-dense of order n in S , then all three conditions are equivalent.

Proof: 1. implies 2.: Suppose that X is dense of order $2n$ and that R is a member of $B(S^n \times S^n)$ disjoint from $X^n \times X^n$. From lemma 3, there are countable subsets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of $S \setminus X$ with

$$R \subset \langle A_1, \ldots, A_n, B_1, \ldots, B_n \rangle .$$

From the relations

$$\langle A_1, \ldots, A_n \rangle \in I(X^n) ,$$

$$\langle B_1, \ldots, B_n \rangle \in I(X^n) ,$$

$$\langle A_1, \ldots, A_n, B_1, \ldots, B_n \rangle = \langle\langle A_1, \ldots, A_n \rangle , \langle B_1, \ldots, B_n \rangle\rangle$$

it follows that R is $\mathcal{I}(X^n)$-reticulate, as desired.

2. implies 3.: Immediate.

3. implies 1.: (Assuming that X is dense of order n): We show first that if $R \in \mathcal{B}(S^n \times S^n)$ is $\mathcal{I}(X^n)$-reticulate, then R is reticulate in $S^{2n} = S^n \times S^n$. So suppose that $R \subset \langle A, B \rangle$, where A and B are members of $\mathcal{I}(X^n)$. Since X is Borel-dense of order n, there are countable subsets $A_1, \dots, A_n$ and $B_1, \dots, B_n$ of S with

$$A \subset \langle A_1, \dots, A_n \rangle ,$$

$$B \subset \langle B_1, \dots, B_n \rangle .$$

Then $R \subset \langle A_1, \dots, A_n, B_1, \dots, B_n \rangle$, as desired.

Now assume that $R \in \mathcal{B}(S^{2n})$ is not reticulate. Then from lemma 2, R contains a thread T. We have proved that T is also an $\mathcal{I}(X^n)$-thread, so that $(X^n \times X^n) \cap T \neq \emptyset$.

Q.E.D.

Proposition: Suppose that $X \subset S$ is Borel-dense (of order 1). Let A be an analytic set. Then for each $n \geq 1$, the following conditions are equivalent:

1) X is Borel-dense of order $2n$;
2) X^n is strongly Blackwell;
3) X^n is Blackwell;
4) $X^n \times S$ is strongly Blackwell;
5) $X^n \times S$ is Blackwell;
6) $X^n \times A$ is strongly Blackwell;
7) $X^n \times A$ is Blackwell.

<u>Demonstration</u>: We prove first that conditions 1, 2, 3 are equivalent <u>via</u> induction on n. The case $n = 1$, which was first proved in [17], may be quickly established by an appeal to lemma 8. We assume the result for n and prove it for $n + 1$.

<u>1. implies 2.</u>: This follows from lemmas 8 and 9.

<u>2. implies 3.</u>: Immediate.

<u>3. implies 1.</u>: Since every relatively Borel subset of a Blackwell space is again Blackwell [1 ; p. 27], it follows that X^n is Blackwell. From the induction hypothesis, we see that X is Borel-dense of order $2n$ (hence also of order $n + 1$). Now from lemma 8, $X^{n+1} \times X^{n+1}$ meets every $\mathcal{I}(X^{n+1})$-thread. Lemma 9 applies to prove that X is Borel-dense of order $2(n + 1)$, as desired.

Thus conditions 1, 2, 3 are equivalent for all n. So we fix $n \geqslant 1$ and establish the other equivalences.

<u>1. implies 4.</u>: Lemma 8 says that it suffices to show that $X^n \times S$ is $\mathcal{I}(X^n \times S)$-dense of order 2 in $S^n \times S$. So suppose that R is a member of $\mathcal{B}(S^n \times S \times S^n \times S)$ contained in $(S^n \times S \times S^n \times S) \setminus (X^n \times S \times X^n \times S)$. Let $f\colon S^n \times S \times S^n \times S \to S^n \times S^n$ be defined as the projection

$$f(s_1, \ldots, s_n, s_{n+1}, t_1, \ldots, t_n, t_{n+1}) = (s_1, \ldots, s_n, t_1, \ldots, t_n).$$

Then $f(R)$ is an analytic subset of $(S^n \times S^n) \setminus (X^n \times X^n)$. Since X is Borel-dense of order $2n$, there are countable subsets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of $S \setminus X$ (lemma 3) with

$$f(R) \subset \langle A_1, \ldots, A_n, B_1, \ldots, B_n \rangle$$

and

$$R \subset \langle\langle A_1, \ldots, A_n, \phi \rangle, \langle B_1, \ldots, B_n, \phi \rangle\rangle .$$

Since $\langle A_1, \ldots, A_n, \phi \rangle$ and $\langle B_1, \ldots, B_n, \phi \rangle$ do not intersect $X^n \times S$, we see that R is $I(X^n \times S)$-reticulate, as desired.

<u>4. implies 5.</u>: Immediate.

<u>4. implies 6.</u>: This follows from the fact that measurable images of strong Blackwell spaces are again strongly Blackwell [1 ; p. 23] . The space $X^n \times A$ is an image of $X^n \times S$.

<u>6. implies 7.</u>: Immediate.

<u>7. implies 3.</u>: Note that X^n is a relative Borel subset of $X^n \times A$.

<u>5. implies 3.</u>: Same as the preceding.

Q.E.D.

Lemma 6 implies the following

<u>Corollary 1</u>: Let $S \setminus X$ have property (s°) . Then X^n and $X^n \times A$ are strongly Blackwell for each $n \geq 1$.

Using lemma 7, we see that for each k, there are sets X which are dense of order $2k$, but not of any higher order. This establishes

<u>Corollary 2</u>: For each k, there is some space X with X^n strongly Blackwell for each $n \leq k$ and not Blackwell for every $n > k$.

Finally, we note that our results serve to answer an open question of D. Ramachandran. In [10], he asks whether regular conditioning of probabilities is always possible on strong Blackwell domains. Were it so, then other results of his would maintain strong Blackwell spaces as a natural foundation for probability theory. Unfortunately, this is not the case:

<u>Corollary 3</u>: There is a strong Blackwell space Y, a Borel probability P on $(Y, \mathcal{B}(Y))$, and a countably generated sub-σ-algebra $\mathcal{C}$ of $\mathcal{B}(Y)$ such that no regular conditional probability of P given $\mathcal{C}$ exists.

<u>Proof</u>: Let X be a subset of S Borel-dense of order 2, but not order 4 (lemma 7). Then the proposition implies that $Y = X \times S$ is strongly Blackwell. But X is not universally measurable (lemma 6), so that Theorem 4 in [13] applies to prove the corollary.

Q.E.D.

<u>Question</u>: How much of the main proposition is retained at $n = \infty$?

§3. References

[1] Bhaskara Rao, K. P. S. and Rao, B. V. : Borel spaces. Dissertationes Mathematicae CXC (1981).

[2] Brown, J. B. and Cox, G. V. : Classical theory of totally imperfect spaces. Real Analysis Exchange 7 (1981-2) 185-232.

[3] Bzyl, W. : On analytic-dense Blackwell sets. (pre-print).

[4] Bzyl, W. and Jasiński, J. : A note on Blackwell spaces. Bull. Acad. Polon. Sci. (to appear).

[5] Fremlin, D. : On Blackwell algebras (unpublished manuscript, 1980).

[6] Fremlin, D. : A very non-Blackwell space (unpublished manuscript, 1983).

[7] Jasiński, J. : On the Blackwell property of Lusin sets (pre-print).

[8] Jasiński, J. : On the combinatorial properties of Blackwell spaces (pre-print).

[9] Marczewski (Szpilrajn), E. : Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles. Fundamenta Mathematicae 24 (1935) 17-34.

[10] Ramachandran, D. : Perfect Measures, Macmillan India, Delhi (1979).

[11] Sarbadhikari, H. : A note on reticulated sets (pre-print).

[12] Shortt, R. M. : Borel-density, the marginal problem and isomorphism types of analytic sets. Pacific Journal of Mathematics 113 (1984) 183-200.

[13] Shortt, R. M. : Universally measurable spaces: an invariance theorem and diverse characterisations. Fund. Math. (in press).

[14] Shortt, R. M. : Minimal complementation and maximal conjugation for partitions with an application to Blackwell sets. Fund. Math. (in press).

[15] Shortt, R. M. : A generalised Mazurkiewicz-Sierpiński theorem. Colloquium Mathematicum (to appear).

[16] Shortt, R. M. : Reticulated sets and the isomorphism of analytic powers. (pre-print).

[17] Shortt, R. M. : Borel-dense Blackwell spaces are strongly Blackwell. (pre-print).

[18] Shortt, R. M. and Bhaskara Rao, K. P. S. : Generalised Lusin sets with the Blackwell property. (pre-print).