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Geometric properties of fractals

Fractal is a general describtive term referring to the subsets of the euclidean n-space which are quite different from smooth curves or surfaces or sets which can be well approximated by them. Typical examples are Cantor-type sets, curves of infinite length having tangent nowhere and surfaces without tangent planes. However, Cantor sets with positive Lebesgue measure should not be considered as fractals since Lebesgue density theorem tells that they can locally be well approximated by balls. The term fractal was introduced by Mandelbrot, who has used them to model various physical phenomena, cf. [MB]. In the following I will explain some of the geometric measure theoretic properties of fractals. An excellent introductory text to this area is Falconer's recent book [FK2], and Federer [FH2] gives a rather complete treatment of the integral-dimensional sets.

1. The structure of integral-dimensional sets

The basic tools for studying fractals are the Hausdorff measures. For $0 \le s \le n$ the s-dimensional Hausdorff measure of a set A in R^n is given by

$$H^s(A) = \liminf_{\delta \downarrow 0} \{ \sum_{i=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i \le \delta \}.$$

Here $\alpha(s)$ is a positive normalization constant, which for integral s is chosen so that the restriction of H^s to any sufficiently smooth s-dimensional surface is the s-dimensional area measure on that surface. In particular, H^n is the Lebesgue measure in R^n. It is not hard to show that H^s is a Borel regular outer measure in R^n, and that for any A in R^n there is a unique number $\dim A$, called the Hausdorff dimension of A, determined by the following properties:

$$H^s(A) = \infty \quad \text{for} \quad s < \dim A,$$

$$H^s(A) = 0 \quad \text{for} \quad s > \dim A.$$

The foundations of geometric measure theory were laid by Besicovitch in 1920's and 30's in [B], where he described to an amazing extent the structure of those H^1 measurable subsets and the plane R^2 which have finite H^1 measure. In [FH1] Federer extended most of Besicovitch's theory to H^m measurable subsets A of R^n with $H^m(A) < \infty$, where m is an integer, $0 < m < n$. The underlying principle in this theory is that such a set A can be decomposed into two parts, $A = B \cup C$, such that the geometric measure theoretic properties of B are similar to those of "nice" m-dimensional surfaces, whereas the properties of C are completely opposite. Thus B is the non-fractal and C the fractal part of A. Besicovitch called these parts regular and irregular, and Federer (H^m,m) rectifiable and purely (H^m,m) unrectifiable. Briefly the main results are the following:

(1) Rectifiability properties: H^m almost all of B can be covered with countably many m-dimensional C^1 (or Lipschitz) submanifolds, but $H^m(M \cap C) = 0$ for any such submanifold.

(2) Density theorems: The upper and lower s-dimensional (spherical) densities of a set $E \subset R^n$ at a point $x \in R^n$ are defined as the upper and lower limits as $r \downarrow 0$ of the ratios

$$H^s(E \cap \{y: |x - y| \leq r\})/\alpha(s)r^s.$$

We denote them by $\overline{D}^s(E,x)$ and $\underline{D}^s(E,x)$. Then

$$\overline{D}^m(B,x) = \underline{D}^m(B,x) = 1 \quad \text{for } H^m \text{ almost all} \quad x \in B,$$

$$\underline{D}^m(C,x) < 1 \quad \text{for } H^m \text{ almost all} \quad x \in C.$$

The latter result was proved in [MJ2] and [MP1]. For $m = 1$, Besicovitch and Moore [ME] have proved the stronger result $\underline{D}^1(C,x) < \overline{D}^1(C,x)$ for H^1 almost all $x \in C$. It is not known whether this holds for $m \geq 2$.

(3) Existence of tangents: At H^m almost every $x \in B$ there is an approximate m-dimensional tangent plane V for B, which means that V is an m-plane containing x such that for every $\delta > 0$

$$\lim_{r \downarrow 0} r^{-m} H^m(B \cap \{y: |x - y| \le r, \text{ dist } (y,V) \ge \delta |x - y|\}) = 0.$$

At H^m almost every $x \in C$ no such tangent plane exists.

(4) Orthogonal projections: Let $G(n,m)$ be the set of all linear m-dimensional subspaces of R^n, and let $P_V: R^n \to V$ be the orthogonal projection onto $V \in G(n,m)$. There is a natural orthogonally invariant probability measure $\gamma_{n,m}$ on $G(n,m)$, which in the cases $m = 1$ and $m = n - 1$ can be identified with the surface measure on the unit sphere of R^n. The results of Besicovitch and Federer give:

$$H^m(P_V B) > 0 \text{ for } \gamma_{n,m} \text{ almost all } V \in G(n,m), \text{ if } H^m(B) > 0,$$

$$H^m(P_V C) = 0 \text{ for } \gamma_{n,m} \text{ almost all } V \in G(n,m).$$

The problem whether the corresponding result holds for sets A having finite integralgeometric measure $I_1^m(A)$ was solved in the negative in [MP5].

2. Densities of general fractals

We now turn to general fractals whose Hausdorff dimension need not be integral. The fundamental work was done by Marstrand in 1954 in [MJ1]. We first consider their densities. The following basic result, essentially due to Besicovitch, is often very useful: If $A \subset R^n$ is H^s measurable with $H^s(A) < \infty$, then

$$2^{-s} \le \overline{D}^s(A,x) \le 1 \text{ for } H^s \text{ almost all } x \in A,$$

$$\overline{D}^s(A,x) = 0 \text{ for } H^s \text{ almost all } x \in R^n \setminus A.$$

It may happen that $\underline{D}^s(A,x) = 0$ everywhere in A although $H^s(A) > 0$. The more delicate density structure of integral-dimensional sets was described in (2). The corresponding problem for non-integral dimensional sets was solved by Marstrand in [MJ3]:

If s is not integral, A is H^s measurable and $H^s(A) < \infty$, then

$$\underline{D}^s(A,x) < \overline{D}^s(A,x) \quad \text{for } H^s \text{ almost all } x \in A.$$

In [MJ1] Marstrand also considered densities in angles (in place of discs) of subsets of the plane. Most of these results were generalized to higher dimensions by Salli in [S]; he also replaced angles by much more general sets.

In the same paper Salli showed that for the so-called self-similar fractals much sharper density theorems hold. A subset of R^n is self-similar if it can be split into a finite number of "essentially disjoint" parts each of them being geometrically similar to the whole set. For a precise definition and elegant theory see Hutchinson's paper [H]. Standard examples are the symmetric Cantor sets. Salli showed that for self-similar sets the upper and lower densities have a constant value almost everywhere in that set. This applies also to the generalized angular densities.

3. Projections of fractals

The basic projection theorem is: If A is a Suslin set in R^n, then

(1) $$\dim A > m \text{ implies } H^m(P_V A) > 0 \text{ for } \gamma_{n,m} \text{ a.a. } V \in G(n,m),$$

(2) $$\dim A \leq m \text{ implies } \dim P_V A = \dim A \text{ for } \gamma_{n,m} \text{ a.a. } V \in G(n,m).$$

The case $m = 1$, $n = 2$ was proved by Marstrand in [MJ1]. In [KR] Kaufman gave new proofs using Fourier transform in (1) and potential theory in (2). Later both Fourier transform and potential theoretic methods have been very useful in several questions on fractals. Kaufmann also showed in the case $m = 1$, $n = 2$ that the set of exceptional directions in (2) has Hausdorff dimension at most $\dim A$, and this bound is sharp, cf. [KM]. Generalizations to higher dimensions as well as in other respects can be found in [MP2] and [FK1]. For example, in [FK1] Falconer derives an upper bound for the Hausdorff dimension of the set of the exceptional directions in (1), which in the case $m = 1$, $n = 2$ equals $2 - \dim A$. Davies has shown that for general non-Suslin sets the above projection theorem fails, see [D].

The results described above give good information about the measures $H^m(P_V A)$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$ in the cases $H^m(A) < \infty$ and $\dim A > m$, but they say nothing about the case where $\dim A = m$ and A has non-σ-finite H^m measure. Recently Falconer and Talagrand [T] have shown that there is hardly anything to say ingeneral. They have shown (independently) that modulo certain measurability assumptions one can give the measures of almost all of the projections in advance, and then find the set. Falconer has gone even further and shown that, again with mild measurability assumptions, given $E_V \subset V \in G(n,m)$ there is $A \subset R^n$ such that $E_V \subset P_V A$ and $H^m(P_V A \setminus E_V) = 0$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$. In the case $m = 1$, $n = 2$ Falconer's result is in [FK2, Chapter 7], and the general version will appear later.

4. Intersections of fractals

Suppose that A and B are Suslin sets in R^n. The basic question is to find relations between the Hausdorff dimensions of A and B and those of the intersections $A \cap fB$, where f runs through the isometry group of R^n, or some other family of transformations. Several results of this type appear in [MP4], and a summary of them was given in [MP3]. There the following two cases were considered: (1) B is "nice" integral-dimensional and f is an isometry. (2) A and B are fractals and f is a similarity. About the same time Kahane [KJ] considered independently the second case with more general affine maps than similarities. One of his results is:

Let G be a closed subgroup of the general linear group of R^n acting transitively in $R^n \setminus \{0\}$, and let τ be an invariant measure in G. If A has positive s-capacity $C_s(A)$ and if $H^t(B) > 0$, then, if $s + t > n$,

$$H^n\{z \in R^n: C_{s+t-n}(A \cap (gB + z)) > 0\} > 0$$

for τ almost all $g \in G$; in particular, $\dim A \cap (gB + z) \geq s + t - n$.

If $0 < H^s(A) < \infty$ and $0 < H^t(B) < \infty$ and if B satisfies some extra conditions, e.g. $\underline{D}^t(B,x) > 0$ for $x \in B$, then one can see as in [MP4] that the latter inequality

$\geq$ can be replaced by equality provided $\dim A + \dim B - n \geq 0$. That some extra conditions are needed follows from the examples constructed in [KJ], [MP4] and [FK4].

It is not clear in what generality the results of the above type hold in R^n, $n \geq 2$, with $G = O(n)$, the orthogonal group of R^n. In R^1, where the rotation group is degenerate, they don't hold at all, see [KJ] or [MP4]. However, it seems that certain estimates obtained by Falconer in [FK3] could be used to improve some of the results of [KJ] and [MP4].

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