Real Analysis Exchange Vol. 8 (1982-83)

Kenneth R. Kellum
Department of Mathematics and Computer Science
San Josè State University
San Josè, California 95192

Almost Continuity and Connectivity--Sometimes It's As Easy to Prove a Stronger Result

This note is an attempt to interest more analysts in almost continuous functions. One reason that an analyst might care about almost continuity is that when he or she wishes to prove that a real function is a connectivity function it is often just as easy (or even easier) to prove a stronger result--that the function is almost continuous. I give two examples, one old and one new. Also I show that some results of Brown on negligible sets for connectivity functions and some recent results of Cristian and Tevy on associated sets easily extend to almost continuous functions.

Unless otherwise noted all functions considered here are real functions defined on the real line, R. No distinction is made between a function and its graph.

That a function f is <u>connected</u> just means that f is a connected set. We say that f is a <u>connectivity</u> function if $f|C$ is connected whenever C is connected. While connected functions are not necessarily connectivity functions in general, the notions are the same for functions from the real line into itself. That f is <u>almost continuous</u> means that if D is an open set such that $f \subseteq D$ then there exists a continuous function g with the same domain as f such that $g \subseteq D$.

It is not hard to see that if $f: R \to R$ is almost continuous then f is connected [11]. The converse, however, is not true (see [8], for example). These classes of functions share a number of properties. They coincide, for instance, for functions of Baire class 1 [1]. Both classes can be characterized in terms of closed subsets of the plane. While it is possible to extend many results about connectivity functions to results about almost continuous functions, extensions of this type are not always possible. For example, Brown in [1] has shown that a Darboux function which is the pointwise limit of a sequence of functions which are continuous on the right must be a connectivity function and that such a function need not be almost continuous.

<u>Proposition 1.</u> [6] In order that $f: R \to R$ be a connectivity function it is necessary and sufficient that whenever M is a continuum (compact connected set) in the plane such that f both has a point directly above some point of M and a point directly

below some point of M, then $f \cap M \neq \emptyset$.

Suppose $f:R \to R$. By a <u>blocking set</u> of f we mean a closed subset K of the plane such that $f \cap K = \emptyset$ and such that if $g:R \to R$ is continuous then $g \cap K \neq \emptyset$. Obviously, f is almost continuous if and only if it has no blocking set. An <u>irreducible blocking set</u> (IBS) K of f is a blocking set of f such that no proper subset of K is a blocking set.

<u>Proposition 2.</u> [8] Suppose $f:R \to R$ fails to be almost continuous. Then there exists an IBS K of f and the x-projection of K is a non-degenerate connected set.

<u>A classic result of Jones</u>

In 1942 Jones [7] gave an example of a connectivity function $f:R \to R$ such that f is a dense subset of the plane and f satisfies $f(x + y) = f(x) + f(y)$. The fuction f is constructed by transfinite induction. In order to make f be connected, Jones insures that if M is a closed subset of the plane with uncountable x-projection, then $M \cap f \neq \emptyset$. In order to convince someone that f is connected one can discuss what sets can separate subsets of the plane and get something akin to Proposition 1. I find it easier to prove Proposition 2 (it's not hard) and show that f is almost continuous.

A recent result of Ceder

The following lemma is an "unbounded" version of the technique used in [10] to produce an almost continuous function which is a dense subset of the unit square.

Let p_1 and p_2 denote, respectively, the projections of the plane onto the x-axis and the y-axis.

Lemma 1. Suppose $f: R \to R$ has the property that if u is an upper semi-continuous function with domain a non-degenerate interval and range a subset of R then $f \cap cl(u) \neq \emptyset$. Then f is almost continuous.

Proof. Assume the contrary and let K be an IBS of f. Note that $p_2(K) = R$, for otherwise K would miss a constant function. By the Baire category theorem, for some integer n, $p_1(K \cap p_2^{-1}([n,n+1]))$ must contain a closed interval, [a,b]. For each $x \in [a,b]$ let $(x,u(x))$ be the highest point of $K \cap ([a,b] \times [n,n+1])$ with abscissa x. Then u is upper semi-continuous, leading to a contradiction.

Recently Ceder [3] gave an example of a connectivity function $f: R \to R$ such that $f|A$ is discontinuous whenever A is uncountable. Using the preceeding lemma one need only examine Ceder's proof to see that the his function is in fact almost continuous. All that one need do is replace the continuum H at the top of page 160 of Ceder's paper with the closure of an upper semi-continuous function.

Note that Jones' example does not depend on the continuum hypothesis while Ceder's construction does use CH.

Results of Brown on negligible sets

Let the letter I denote the interval [0,1]. The following results are stated for functions from I into itself because that is the convention in the references. These results also hold for functions from R into R.

Suppose $f: I \to I$ is almost continuous and that $M \subseteq I$. We say that M is f-negligible if every function $g: I \to I$ which agrees with f on $I \setminus M$ is almost continuous. (Note that my use of the term "negligible" is different than Brown's). The following two theorems are simply restatements of Theorems 1 and 2 of [2], with "connectivity" replaced with "almost continuous".

Theorem 1. If M is a subset of I, then there exists an almost continuous function $g: I \to I$ such that M is g-negligible if and only if $I \setminus M$ is c-dense in I.

Proof. Just use Brown's proof together with Lemma 1 and the fact that an almost continuous function from

I into I is connectivity.

Theorem 2. If $g: I \to I$ is almost continuous then the following statements are equivalent to each other:

(i) g is dense in I^2.

(ii) Every nowhere dense subset of I is g-negligible.

(iii) There exists a dense subset of I which is g-negligible.

Proof. To show that (i) implies (ii), suppose $g: I \to I$ is almost continuous and dense, M is nowhere dense and $f: I \to I$ agrees with g on $I \setminus M$. Assume that f is not almost continuous and let Q be an IBS of f. Since $p_1(Q)$ is an interval, $\mathrm{int}(p_1(Q))$ must contain a point z of $I \setminus \mathrm{cl}(M)$. By the irreducibility of Q there exist continuous functions $s, t: I \to I$ such that $s|[0,z] \cap Q = \phi = t|[z,1] \cap Q$. Let A and B be, respectively, circular neighborhoods of $(z, s(z))$ and $(z, t(z))$ such that $(A \cup B) \cap Q = \emptyset$. Since g is dense there exist c and d such that $c < z < d$, $(c, g(c)) \in A$, $(d, g(d)) \in B$ and $[c,d] \subseteq I \setminus M$. Let $P =$

$$((\{c\} \times I) \setminus A) \cup ((\{d\} \times I) \setminus B) \cup (p_1^{-1}([c,d]) \cap Q).$$

Since $g \cap P = \emptyset$, there is a continuous function $h: I \to I$ which misses P. We can now obtain a contradiction by constructing a continuous function from I into I which misses Q. Do this by taking parts of

$s|[0,c]$, $t|[d,1]$ and $h|[c,d]$ and joining them with linear functions inside A and B.

That (ii) implies (i) and that (iii) implies (i) follow exactly as in Brown's proof. The proof that (i) implies (iii) given below differs only superficially from Brown's.

Suppose again that $g: I \to I$ is almost continuous and dense. Then $g^{-1}(1/2)$ is easily seen to be dense in I. Let $g^{-1}(1/2) = M \cap K$, where M and K are disjoint and dense in I. We show that M is g-negligible. To see this, suppose $f: I \to I$ agrees with g on $I \setminus M$. Assume that f is not almost continuous and let Q be a blocking set of f. Since K is dense in I the set $N = \{x \in M : (x,1/2) \in Q\}$ is nowhere dense. But the function $h: I \to I$ which agrees with f on N and agrees with g on $I \setminus M$ misses Q and cannot be almost continuous. Thus N is a nowhere dense set which is not g-negligible. This contradiction completes the proof.

Results of Cristian and Tevy on associated sets

Theorems 3 and 4 and Corollary 1 below are restatements of results of Cristian and Tevy [5], with "almost continuous" replacing "connected". Using Theorem 2 of the present paper, the proofs are virtually identical to those in [5], with "continuum" being replaced by "IBS", and are omitted. Definitions of terms used but not defined here may be found in [5].

Theorem 3. Suppose $f,g:I \to I$ such that f is Darboux, dense and not almost continuous and there exists a finite set $A \subsetneq (0,1)$ such that $f^{-1}(y) = g^{-1}(y)$ for any $y \in I \setminus A$. Then g is dense and not almost continuous.

Corollary 1. Suppose $f,g:I \to I$ are both Darboux and dense and there exists a finite subset A of $(0,1)$ such that $f^{-1}(y) = g^{-1}(y)$ for each $y \in I \setminus A$. Then f and g are both almost continuous or both not almost continuous.

The set A in the two preceeding results cannot in general be replaced either with a countable set or with a nowhere dense set.

Theorem 4. The class of almost continuous functions is not characterizable by associated sets.

Problem. It is known that any function from R to R is the pointwise limit of a sequence of almost continuous functions [9]. However, no intrinsic characterization has been given either for the uniform limit of a sequence of almost continuous or of a sequence of connectivity functions. Bruckner and Ceder [3] have considered the latter problem.

References

1. J. B. Brown, Almost Continuous Darboux Functions and Reed's Pointwise Convergence Criteria, Fund. Math., 86 (1974), 249-263.

2. J. B. Brown, Negligible Sets for Real Connectivity Functions , Proc. Amer. Math. Soc., 24 (1970), 2, 263-269.

3. A. M. Bruckner and J. Ceder, On Jumping Funcions by Connected Sets, Czec. Math. J. 22 (1972), 435-448.

4. J. Ceder, Some Examples on Continuous Restrictions, Real Analysis Exchange, 7 (1981-82), 155-162.

5. B. Cristian and I. Tevy, On Characterizing Connected Functions, Real Analysis Exchange, 6 (1980-81), 203-207.

6. B. D. Garrett, D. Nelms and K. R. Kellum, Characterizations of Connected Real Functions, Jber. Deutsch. Math. Verein, 73 (1971), 131-137.

7. F. B. Jones, Connected and Disconnected Plane Sets and the Functional Equation $f(x) + f(y) = f(x + y)$, Bull. Amer. Math. Soc., 48 (1942), 115-120.

8. F. B. Jones and E. S. Thomas, Connected G_δ Graphs, Duke Math. J., 33 (1966), 341-346.

9. K. R. Kellum, Sums and Limits of Almost Continuous Functions, Colloq. Math., 31 (1974), 125-128.

0. K. R. Kellum and B. D. Garrett, Almost Continuous Real Functions, Proc. Amer. Math. Soc., 33 (1972), 181-184.

.1. J. Stallings, Fixed Point Theorems for Connectivity Maps, Fund. Math., 47 (1959), 249-263.

Received August 13, 1982