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# On Absolute Peano Derivatives

## 1. Introduction

In the recent survey article on Peano derivatives [1], M. J. Evans and C. E. Weil have stated that it is not known whether Laczkovich's absolute Peano derivatives have the -M,M property, the Zahorski property, or property Z. With some further observations on the generalized Peano derivatives studied by the author [3], we show that, in particular, the absolute Peano derivatives do have those properties.

Terminology and notations are those used in the survey article [1] unless otherwise stated. The letter n will be a positive integer throughout the paper.

Now, we review the study in [3]. The (ordinary) $n^{th}$ Peano derivative of f at t is denoted as $f_{(n)}(t)$, the same as that in [1] except that the parentheses are put around n. The generalized $n^{th}$ Peano derivative of f at t as defined in [3], denoted as $f_{[n]}(t)$ with brackets around n, is just the ordinary $(n+k)^{th}$ Peano derivative $g_{(n+k)}(t)$, where g is a $k^{th}$ primitive of f in a neighborhood of t, assuming that f is continuous in that neighborhood and that there exist such k and g for which $g_{(n+k)}(t)$ exists. Note that $f_{[n]}(t)$, if it exists, is unambiguously defined since it is independent of the k and g above. Also note that it might happen that one of $f_{(n)}(t)$ and $f_{[n]}(t)$ exists while the other does not exist. However, if f is assumed to be continuous in a whole neighborhood of t, then the existence of $f_{(n)}(t)$ implies that $f_{[n]}(t)$ exists and equals $f_{(n)}(t)$. In particular, if $f_{(n)}$ exists and $f_{(1)}$ is finite in an interval, then $f_{[n]}$ exists and equals $f_{(n)}$ in that interval. The above statement fails to hold when the parentheses and brackets are interchanged. In fact, even more can be said. It can happen that $f_{[n]}$ exists and is finite on an interval while $f_{[n]}$ is not of the form $g_{(i)}$ on the interval for any integer i. This follows from the following two facts: (i) for each Laczkovich absolute Peano derivative $f^*$ on a compact interval there exist a function g and a positive integer i such that $f^* = g_{[i]}$ on the interval (cf. corollary to theorem 7 in [3]); (ii) there exists a function f such that $f^*$ exists on an interval while $f^*$ is not an (ordinary) $j^{th}$ Peano derivative on that interval for any integer j (cf. theorem 9 in [2]). Thus, the following results proved in [3] are genuine generalizations of those known for the exact Peano derivatives.

<u>Theorem A</u>. Let f be a function defined on [a,b] such that $f_{[n]}$ exists and is finite on [a,b]. Then in the interval [a,b] one has the following:

(I) $f_{[n]}$ is of Baire class one;

(II) $f_{[n]}$ has the Darboux property;

(III) $f_{[n]}$ has the Denjoy property;

(IV) if $f_{[n]}$ is bounded above or below in a non-degenerated interval, then $f^{(n)}$ exists and equals $f_{[n]}$ there.

Here we will show further that $f_{[n]}$ has the -M,M property (cf. the O'Malley-Weil's Oscillation Theorem in section 2), the Zahorski property and property Z (cf. Theorem 4 and its corollary in section 3). It then follows from (i) stated above that every Laczkovich's absolute Peano derivative has those properties, too.

It is helpful to note that if $f_{[n]}(t)$ exists and if g is a $k^{th}$ primitive of f in a neighborhood of t, then $g_{[n+k]}(t)$ exists and equals $f_{[n]}(t)$.

2. The -M,M property.

We state the -M,M property for the generalized Peano derivatives as <u>O'Malley-Weil's Oscillation Theorem</u>: Suppose that $f_{[n]}$ exists and is finite on the interval $I_0$, and let $M \geq 0$. If $f_{[n]}$ attains both M and -M on $I_0$, then there exists a subinterval I of $I_0$ such that $f_{[n]} = f^{(n)}$ on I and $f^{(n)}$ attains both M and -M on I.

For the (ordinary) Peano derivative $f_{(n)}$ and for the approximate derivative $f'_{ap}$, the results were proved by O'Malley and Weil in [6]. Their long and intricate proof for the case $f_{(n)}$ can be carried over for the generalized $f_{[n]}$ and will not be reproduced here. However, to make their argument work for the generalized $f_{[n]}$, besides those results stated as Theorem A in the introduction, we need to prove the following two results.

<u>Theorem 1</u>. Suppose that $f_{[n-1]}$ exists and is finite on [a,b] and $f_{[n]}$ exists and is finite nearly everywhere on [a,b]. If $f_{[n]}$ is Lebesgue summable on [a,b], then $f_{[n-1]}$ is absolutely continuous on [a,b] and

$$f_{[n-1]}(b) - f_{[n-1]}(a) = \int_a^b f_{[n]}(t)dt.$$

<u>Theorem 2</u>. Suppose that $f_{[n-1]}$ exists and is finite on a neighborhood of t. If $f_{[n]}(t)$ exists and is finite, then for each positive number $\varepsilon$ (say, $< 1$) there exists a positive number $\delta$ such that for any x with $0 < |x-t| < \delta/2$ there exists $x_1$ and $x_2$ with $x_1 < x < x_2$ and satisfying the following properties:

(1) $|x_i-x| < \varepsilon|x-t|$ for $i = 1,2$,

and

(2) $|f_{[n-1]}(z) - f_{[n-1]}(t) - (z-t)f_{[n]}(t)| < \varepsilon|z-t|$

for z in a set of positive measure contained in $[x_1,x]$, and in a set of positive measure contained in $[x,x_2]$.

Note that here for convenience we use the convention that "$f_{[0]}(t)$ exists and is finite" means "$f = f_{[0]}$ is continuous not only at t but also in a neighborhood of t." Also, if the domain of the function concerned is a compact interval, then only the suitable one-sided case should be understood at each of the end points of the interval.

Theorem 2, being closely related to Weil's property Z, will be considered in section 3. Here we will give theorem 1 a proof. First, we need refinements of some results on ordinary Peano derivatives due to Sargent in [8]. To be precise, we will start with some convenient notations.

Let m be a fixed non-negative integer, and let F be a function such that $F_{[m]}(t)$ exists and is finite. (Note that for $m = 0$ this means that F is continuous on a neighborhood of t by our convention.) Then define the $m^{th}$ approximation polynomial in x at t as

$$P_m(x) = P_m(F;t;x) = \Sigma_{i=0}^{m}(x-t)^i F_{[i]}(t)/i!,$$

and the $m^{th}$ remainder as

$$R_m(x) = R_m(F;t;x) = F(x) - P_m(x)$$

for any x in the domain of F. Furthermore, the (generalized) $m^{th}$ Peano 0-difference of F from t to x is defined as

$$\varepsilon_m^0(x) = \varepsilon_m^0(F;t;x) = R_m(x)m!/(x-t)^m$$

when $x \neq t$, and $\varepsilon_m^0(t) = 0$. It is clear that $\varepsilon_m^0$ is continuous at t when and only when $F_{[m]}(t)$ is $F_{(m)}(t)$. In general, it may happen that even

$\lim_{x\to t}\varepsilon_m^0(x)$ does not exist. Thus, we need to generalize the 0-difference for our general case. First, we prove the following simple fact.

Lemma 1. Let $F_{[m]}(t)$ exist and be finite, k a positive integer. Then for any two $k^{th}$ primitives, G and H, of F in a neighborhood of t, one has

$$\varepsilon^0_{m+k}(G;t;x) = \varepsilon^0_{m+k}(H;t;x)$$

for all x in that neighborhood.

Proof. Note first that it follows from the definition of the generalized Peano derivatives that

$$G_{[k+i]}(t) = H_{[k+i]}(t) = F_{[i]}(t)$$

for i = 0,1,2, •••, m. Hence

$$R_{m+k}(G;t;x) - R_{m+k}(H;t;x) = R_k(G-H;t;x).$$

As $G^{(k)} = H^{(k)} = F$ on a neighborhood of t, (G-H)(x) is a polynomial in x of degree less than k. Hence $R_k(G-H;t;x) = 0$ for x in that neighborhood by the Taylor remainder theorem. Then

$$R_{m+k}(G;t;x) = R_{m+k}(H;t;x)$$

whence

$$\varepsilon^0_{m+k}(G;t;x) = \varepsilon^0_{m+k}(H;t;x).$$

The number $\varepsilon^{0}_{m+k}(G;t;x)$ or $\varepsilon^{0}_{m+k}(H;t;x)$ in lemma 1 is then unambiguously denoted as $\varepsilon^{k}_{m}(F;t;x)$ and is called the $m^{th}$ Peano k-difference of F from t to x. Note that it follows from the definitions involved that if $F_{[m]}(t)$ exists and is finite, then there exists an integer k such that $\varepsilon^{k}_{m}(F;t;x)$, as a function of x, is continuous at x = t (i.e. $\lim_{x\to t}\varepsilon^{k}_{m}(F;t;x) = 0$), and furthermore $F_{[m+1]}(t)$ exists if and only if there exists a non-negative integer k such that

$$\lim_{x\to t} \varepsilon^{k}_{m}(F;t;x)\,(m+k+1)/(x-t)$$

exists and in this case this limit is $F_{[m+1]}(t)$.

Now, we come to prove the following basic result.

Lemma 2. For any pair of non-negative integers m,k, there exists a positive number $A^{k}_{m}$ such that

$$(3) \qquad |F_{[m]}(d)-F_{[m]}(c)| \leq A^{k}_{m}\ \omega^{k}_{m}(F;[c,d])$$

whenever F is a continuous function defined on the interval [c,d] such that $F_{[m]}(c)$ and $F_{[m]}(d)$ exist and are finite, where $\omega^{k}_{m}(F;[c,d])$ is defined to be the maximum of $\{ \sup_{c\leq x\leq d} |\varepsilon^{k}_{m}(F;c;x)|, \sup_{c\leq x\leq d} |\varepsilon^{k}_{m}(F;d;x)|\}$.

Proof. The assertion is trivial when both m and k are zero by taking $A^{0}_{0}$ to be any real number not less than 1. Thus, we assume that $m + k \geq 1$.

Let F be any continuous function such that $\omega^{k}_{m}(F;[c,d])$ makes sense and (without loss of generality) is less than $+\infty$, and let G be a $k^{th}$ primitive of F on [c,d]. (Note that for k = 0 this means that G = F + C on [c,d] for some constant C.) For convenience, denote n = m + k, and we will prove the following equality:

$$(4) \qquad G_{[n]}(d) - G_{[n]}(c) = \Sigma_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{r^n}{n!}[\varepsilon_n(c;r)-\varepsilon_n(d;r)],$$

where

$$\varepsilon_n(c;r) = \varepsilon_n^0(G;c;c+rh)$$

$$\varepsilon_n(d;r) = \varepsilon_n^0(G;d;d-rh)$$

with $h = (d-c)/n$. Note that

$$\varepsilon_n(c;0) = \varepsilon_n(d;0) = 0,$$

and for $0 < r \leq n$,

$$\varepsilon_n(c;r) = [G(c+rh)-\Sigma_{i=0}^{n}(rh)^i G_{[i]}(c)/i!]n!/(rh)^n,$$

$$\varepsilon_n(d;r) = [G(d-rh)-\Sigma_{i=0}^{n}(-rh)^i G_{[i]}(d)/i!]n!/(-rh)^n.$$

Since $h = (d-c)/n$ and $\binom{n}{r} = \binom{n}{n-r}$ for $0 \leq r \leq n$, one has

$$\Sigma_{r=0}^{n}(-1)^{n-r}\binom{n}{r}[G(c+rh)-(-1)^n G(d-rh)] = 0.$$

Hence the sum on the right hand side of (4) is equal to

$$R = \frac{1}{h^n}\Sigma_{r=0}^{n}\{(-1)^{n-r}\binom{n}{r}\Sigma_{i=0}^{n}[(-1)^{i+n}G_{[i]}(d)-G_{[i]}(c)]h^i r^i/i!\}.$$

By interchanging the summation order, one has

$$R = \frac{1}{h^n} \Sigma_{i=0}^{n} \{B_n^i [(-1)^{i+n} G_{[i]}(d) - G_{[i]}(c)] h^i / i!\},$$

where

$$B_n^i = \Sigma_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^i \qquad \text{(for } i = 0,1,2, \cdots, n\text{)}$$

and is known to be equal to n! when i = n, and to 0 when i = 0,1,2, ···, n-1. Hence one has

$$R = \frac{1}{h^n} B_n^n [(-1)^{2n} G_{[n]}(d) - G_{[n]}(c)] h^n / n!$$

$$= G_{[n]}(d) - G_{[n]}(c), \text{ the left hand side of (4).}$$

Note that $G_{[n]}(\alpha) = F_{[m]}(\alpha)$ for $\alpha = c$ and for $\alpha = d$, and

$$\varepsilon_n(c;r) = \varepsilon_m^k(F;c;c+rh),$$

$$\varepsilon_n(d;r) = \varepsilon_m^k(F;d;d-rh).$$

Hence, by applying the triangular inequality, one obtains from the equality (4) the required inequality (3) by taking $A_m^k = 2\Sigma_{r=0}^{n} \binom{n}{r} r^n / n!$, which depends only on the integer n = m + k. The proof is hence completed.

Theorem 3. Let $m \geq 0$ and let F be a function such that $F_{[m]}$ exists and is finite on [a,b]. If $F_{[m+1]}$ exists and is finite nearly everywhere on a set $E \subset [a,b]$, then $F_{[m]}$ is (ACG) on E (by which we mean that E can be written as a union of countably many sets on each the function $F_{[m]}$ is absolutely continuous in the wide sense, i.e. AC as given Saks' book [7]).

Proof. Since a function is AC on every singleton set, one assumes, without loss of generality, that $F_{[m+1]}$ exists and is finite everywhere on the set E. For each positive integer k, let $E^k$ be the set of all t in E such that

$$F_{[m+1]}(t) = \lim_{x \to t} \varepsilon_m^k(F;t;x)(m+k+1)/(x-t),$$

and for each positive integer p and each integer i, let $E_{p,i}^k$ be the set of all t in $E^k \cap [\frac{i-1}{p}, \frac{i}{p}]$
such that

$$|\varepsilon_m^k(F;t;t+h)| < p|h|$$

whenever t + h is in [a,b] with $0 < |h| \leq 1/p$. Then one sees easily that $E = \cup E_{p,i}^k$, where the union is taken through all integers k, p, i involved. It suffices to show that $F_{[m]}$ is AC on each $E_{p,i}^k$. To see this, let c,d be in $E_{p,i}^k$ with $c < d$. Then, for $\alpha = c$ and d, one has

$$|\varepsilon_m^k(F;\alpha;x)| < p|x-\alpha| \leq p(d-c)$$

for all x in (c,d), and hence

$$\omega_m^k(F;[c,d]) \leq p(d-c).$$

Thus, by lemma 2, we conclude that

$$|F_{[m]}(d)-F_{[m]}(c)| \leq pA_m^k|d-c|$$

for all c,d in $E_{p,i}^k$, and hence it follows that $F_{[m]}$ is AC on $E_{p,i}^k$, completing the proof.

Lemma 2 and theorem 3 are essentially due to Sargent in [8] for the ordinary Peano derivatives on intervals. Our proof is a simple adaptation to our hypothesis of the proof given in [8]. The details in proving the equality (4) were carried out here to indicate the algebraic identities involved. To prove theorem 1, we do not need any more refinements of the other results in [8]. Instead, we need the following result.

Lemma 3. Let G be a function which is of Baire class one, has the Darboux property and satisfies Lusin's condition (N) on [a,b]. Then G is a constant on [a,b] provided that $G'(x) = 0$ for almost all x at which $G'(x)$ exists and is finite.

If G is continuous, this lemma is a consequence of the monotonicity theorem in Saks' book [7], page 282. The general case follows from a generalization of that monotonicity theorem recently presented in [4]. We omit the details here. Now, we are in a position to give a proof for theorem 1.

Proof of Theorem 1. Let $f_{[n]}$ be Lebesgue summable on [a,b], and let K be an indefinite Lebesgue integral of $f_{[n]}$ on [a,b]. Then K is AC on [a,b] and $K' = f_{[n]}$ almost everywhere in [a,b]. Note that $f_{[n-1]}$ is of Baire class one, has the Darboux property by (I) and (II) in Theorem A, and is (ACG) by Theorem 3. Hence the function $G = K - f_{[n-1]}$ also has all these properties since K is AC on [a,b]. It is well-known that a continuous (ACG) function (i.e. an ACG function as defined in Saks' book) has Lusin's property (N). A simple check of the proof in Saks' book [7], page 225, shows that the continuity is superfluous. Hence our (ACG) function G here also satisfies Lusin's condition (N) on [a,b]. Now, we show further that $G'(x) = 0$ for almost all x at which $G'(x)$ exists and is finite. To be precise, let

$$S = \{x : x \text{ in } [a,b] \text{ and } G'(x) \text{ exists and is finite}\}$$

$$T = \{x : x \text{ in } [a,b] \text{ and } K'(x) \text{ exists, is finite, and equals } f_{[n]}(x)\}$$

$$U = S \cap T.$$

Then $|U| = |S|$ since $|T| = b - a$. Let x be in U. One has that both $G'(x)$ and $K'(x)$ exist and are finite, so that $(f_{[n-1]})'(x)$ exists and

$$\begin{aligned} G'(x) &= K'(x) - (f_{[n-1]})'(x) \\ &= f_{[n]}(x) - (f_{[n-1]})'(x). \end{aligned}$$

But then one shows that $(f_{[n-1]})'(x) = f_{[n]}(x)$, so that we have $G'(x) = 0$ for all x in U. Therefore, it follows from lemma 3 that G is constant on [a,b] and hence the conclusion of theorem 3 follows.

## 3. Property Z.

We will call a function having property Z a Weil function. Thus, g is a Weil function on an interval I if for each $\varepsilon > 0$ and for each t in I one has

$$(5) \qquad \lim_{i\to\infty} |\{x : x \text{ in } I_i \text{ and } |g(x)-g(t)| \geq \varepsilon\}| / [|I_i| + \text{dist}(t, I_i)] = 0$$

whenever $\{I_i\}$ is a sequence of subintervals of I and $g \geq g(t)$ on $I_i$ or $g \leq g(t)$ on $I_i$ for each i, and $\{I_i\}$ converges to t as $i \to \infty$.

<u>Theorem 4</u>. Let f be a function such that $f_{[n]}$ exists and is finite on an interval I. Then $g = f_{[n]}$ is a Weil function on I.

When $g = f_{(n)}$ or $f'_{ap}$, the result was proved by C. E. Weil in [9]. A proof for the case $g = f_{[n]}$ can be obtained by following the line of proof given there for the case $g = f_{(n)}$, using (IV) listed in Theorem A here in the introduction and the lemma in [9]. Here, we will give a proof using the following result due to J. Mařík in [5], which he has used to give a proof for B. S. Babcock's result that approximate Peano derivatives are Weil's functions.

<u>Theorem B</u>. (Theorem 1, Mařík [5]). Let m be a positive integer and let G be a function defined in a neighborhood of the point t such that the $m^{th}$ approximate Peano derivative $G_{(m),ap}(t)$ exists and is finite, and define $Q(x) = \Sigma_{\ell=0}^{m}(x-t)^{\ell}G_{(\ell),ap}(t)/\ell!$ for all real number x. Let $\varepsilon > 0$ and $\eta > 0$. Then there exists a $\delta > 0$ with the following properties:

(a) If J is a subinterval of $(t-\delta, t+\delta)$, j an integer with $0 < j \leq m$ and if either $G^{(j)} \leq Q^{(j)}$ on J or $G^{(j)} \geq Q^{(j)}$ on J, then

$$|\{x: x \text{ in } J \text{ and } |G^{(j)}(x)-Q^{(j)}(x)| \geq \varepsilon|x-t|^{m-j}\}| \leq \eta[|J|+\text{dist}(t,J)]. \tag{6}$$

(b) If J is any subinterval of $(t-\delta, t+\delta)$, then the inequality (6) holds for $j = 0$.

Proof of Theorem 4. Let $\varepsilon > 0$ and $t$ in $I$ be fixed, and let $\{I_i\}$ be a sequence of intervals satisfying the conditions described in the definition of a Weil function. To prove (5), it suffices to show that for each $\eta > 0$, there exists an $i_0$ such that

$$(7) \qquad |\{x : x \text{ in } I_i \text{ and} |g(x)-g(t)| \geq \varepsilon\}| \leq \eta(|I_i|+\text{dist}(t,I_i))$$

whenever $i \geq i_0$. Now, note that since $g(t) = f_{[n]}(t)$, there exists a positive integer $k$ such that for a $k^{th}$ primitive $G$ of $f$ in a neighborhood of $t$, one has $f_{[n]}(t) = G_{(n+k)}(t)$ $(= G_{(n+k),ap}(t))$. Thus, applying theorem B to the function $G$ with $m = n + k$ we have a $\delta > 0$ such that properties (a) and (b) hold. Since $f_{[n]} \geq f_{[n]}(t)$ on $I_i$ or $f_{[n]} \leq f_{[n]}(t)$ on $I_i$ implies that $f^{(n)} = f_{[n]}$ on $I_i$ by (IV) in theorem A, and since $f^{(n)} = G^{(n+k)}$ wherever $f^{(n)}$ exists, it follows from (6) with $j = m$ that (7) holds when $i_0$ is large enough to insure that $I_i \subset (t-\delta, t+\delta)$ for all $i \geq i_0$. Such an $i_0$ exists because $\{I_i\}$ converges to $t$ as $i\to\infty$. The proof is hence completed.

Corollary. If $f_{[n]}$ exists and is finite on an interval, then $g = f_{[n]}$ is a Zahorski function there, i.e. for each open interval $I$ and for each $t$ in $g^{-1}(I)$ one has

$$\lim_{i\to\infty}|I_i|/\text{dist}(t,I_i) = 0$$

whenever $\{I_i\}$ is a sequence of intervals converging to $t$ and $|I_i \cap g^{-1}(I)| = 0$ for each $i$. (Note that the last equality was missed in the definition 2.1 in [1].)

Proof. This follows from theorem 4, and (II) and (III) in theorem A since a Weil function with both Darboux and Denjoy properties must be a Zahorski function (see the Remark and the Theorem in [9]).

Following Mařík's step once more, we use theorem B to prove the following result (cf. theorem 3 in [5], where the case for approximate Peano derivative was considered), of which theorem 2 in section 2 is a simple corollary.

Theorem 5. Let f be a function defined on a neighborhood of the point t such that $f_{[n]}(t)$ exists and is finite, and let $0 \leq i \leq n$. Then for each $\varepsilon > 0$ and each $\eta^* > 0$ there exists a $\delta > 0$ with the following property: If L is a subinterval of $(t-\delta, t+\delta)$ such that $f_{[i]}$ exists and is finite on L and that

$$|f_{[i]}(x)-P_n^{(i)}(x)| \geq \varepsilon|x-t|^{n-i}$$

for almost all x in L, then

$$|L| \leq \eta^* \text{dist}(t,L),$$

where $P_n(x) = P_n(f;t;x) = \Sigma_{\ell=0}^{n}(x-t)^{\ell}f_{[\ell]}(t)/\ell!$, and, of course, $P_n^{(i)}$ denotes the ordinary $i^{th}$ derivative of $P_n$.

Proof. Let G be as in the proof of theorem 4, i.e. $G^{(k)} = f$ on a neighborhood of t and $f_{[n]}(t) = G_{(n+k)}(t)$. Applying theorem B to the function G with $m = n + k$, and with $\eta = \eta^*/(1+\eta^*)$, one obtains a $\delta > 0$ having the properties stated there. Let L be an interval as stated above. Note that $G_{[\ell+k]}(t) =$

$f_{[\ell]}(t)$ for $\ell = 0, 1, 2, \cdots, n$, so that one has $P_n^{(i)}(x) = Q^{(i+k)}(x)$. Now, since $G_{[i+k]} = f_{[i]}$ wherever $f_{[i]}$ exists, one has

$$|G_{[i+k]}(x) - Q^{(i+k)}(x)| \geq \varepsilon|x-t|^{n-i} = \varepsilon|x-t|^{m-(i+k)}$$

almost everywhere on L. Now, if $L\cap(t,\infty)$ is non-empty, set $J = L\cap(t,\infty)$, and otherwise set $J = L\cap(-\infty,t)$. Then, using the Denjoy property and the Darboux property of the function $(G-Q)_{[i+k]}$ and following the argument in the proof of lemma 4 in [5], one proves that $G^{(i+k)}$ exists and either $G^{(i+k)} > Q^{(i+k)}$ on J or $G^{(i+k)} < Q^{(i+k)}$ on J. Hence, by the inequality (6) with $j = i + k$, we have $|J| \leq \eta(|J|+\text{dist}(t,J))$ whence $|J| \leq \eta^*\text{dist}(t,J)$. In particular, $\text{dist}(t,J) > 0$ so that $J = L$. Thus $|L| \leq \eta^*\text{dist}(t,J)$, completing the proof.

We end this note by giving a

Proof of Theorem 2: Applying Theorem 5 with $\eta^* = \varepsilon/2$, $i = n - 1$, one obtains a $\delta > 0$ satisfying the property stated there, and such that $f_{[n-1]}$ exists and is finite on $[t-\delta, t+\delta]$. Let x be such that $0 < |x-t| < \delta/2$. Then we will show that there exist $x_1$ and $x_2$ with $|x_i-t| < \delta$ for $i = 1, 2$, and $x_1 < x < x_2$ such that the property (1) (stated in theorem 2) holds and also $|L| > \frac{\varepsilon}{2}\text{dist}(t,L) = \eta^*\text{dist}(t,L)$ for $L = [x_1,x]$ and for $L = [x,x_2]$. Hence it is impossible to have $|f_{[n-1]}(z)-P_n^{(n-1)}(z)| \geq \varepsilon|z-t|$ for almost all z in L, so that the property (2) (stated in Theorem 2) must hold, too. Thus, it remains to show the existence of $x_1$ and $x_2$. For definiteness, let $0 < x - t < \delta/2$. (The argument for the other case $-\delta/2 < x - t < 0$ is similar.) Then the existence of $x_1$ follows immediately by solving for $x_1$ the following inequalities $x - x_1 < \varepsilon(x-t)$ and $x - x_1 > \frac{\varepsilon}{2}(x_1-t)$. The solution set is the non-empty interval $(\varepsilon t+(1-\varepsilon)x, \frac{\varepsilon}{2+\varepsilon}t + \frac{2}{2+\varepsilon}x)$ included in $(t,x)$. Similarly, $x_2$ can be any number in the non-empty interval

$((1-\varepsilon)x + \varepsilon(2x-t),\ (1-\frac{\varepsilon}{2})x + \frac{\varepsilon}{2}(2x-t))$ (included in $(x, 2x-t)$) which is the solution for $x_2$ of the inequalities $x_2 - x < \varepsilon(x-t)$ and $x_2 - x > \frac{\varepsilon}{2}(x-t)$.

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