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Intersections of Sets in n-space.

Let A and B be Borel sets in the Euclidean n-space R^n. We shall all the time allow at least A, often also B, be otherwise completely general. So A might be a Cantor set , the graph of a nowhere differentiable function, etc. Our principal problem is the following: If we move around B with the isometries of R^n, are the Hausdorff measures and Hausdorff dimensions of A and B somehow related to those of the intersections $A \cap fB$? If not, can we say something if we allow f to vary in a larger class of transformations, for example in the group of similarity maps of R^n? We shall give a summary on some results on this problem. The details will appear elsewhere.

The first result of this type is due to J.M.Marstrand [2]. He proved that if $A \subset R^2$ has positive and finite s dimensional Hausdorff measure $H^s(A)$, where $1 < s < 2$, then for H^s almost all $x \in A$ the Hausdorff dimension $\dim A \cap l = s - 1$ and $H^{s-1}(A \cap l) < \infty$ for almost all lines l through x. He also constructed an example where $H^{s-1}(A \cap l) = 0$ for almost all lines through any point of A. Marstrand's theorem was generalized to subsets of R^n with lines replaced by m planes in [3].

A potentialtheoretic approach to this problem was presented in [4]. It was shown that by replacing the Hausdorff measure H^s by the usual potentialtheoretic capacity corresponding to the kernel $|x - y|^{-s}$, one obtains more precise results. For example, if $n-m < s < n$ then for C_s almost all $x \in A$, $C_{s+m-n}(A \cap V) > 0$ for almost all m planes V through x, and

$$bC_s(A) \leq \int C_{s+m-n}(A \cap V)\,d\lambda_{n,m}V \leq cC_s(A),$$

where b and c are positive constants independent of A and $\lambda_{n,m}$ is the usual isometry invariant measure on the space of m planes of R^n. The right hand inequality was proved in [5]. Similar results also hold when the intersections $A \cap V$ are replaced by $A \cap fB$ where f runs through the isometry group of R^n and B is sufficiently nice, e.g. an m dimensional C^1 submanifold of R^n or m rectifiable, i.e. a Lipschitz image of a bounded subset of R^m.

But what if B is not nice at all? Then there is not much to say if we stick to isometries. For example, for any s, $0 \leq s \leq 1$, one can find subsets A and B of R^1 such that $\dim A = \dim B = s$ and for which $A \cap (B + z)$ contains at most one point for any $z \in R^1$, and another pair of such sets for which $\dim A \cap (B + z) = s$ for all $z \in R^1$. However, if we also use homotheties in addition to translations and rotations, that is, if we replace the isometry group by the similarity group, we can again say a lot.

To state the main results we need some notation. Let $\mathcal{L}^n$ be the Lebesgue measure on R^n, θ_n the Haar measure on the orthogonal group $O(n)$ of R^n, and denote the translations and dilations by

$$\tau_z : R^n \to R^n,\ \tau_z(x) = x + z, \text{ for } z \in R^n,$$

$$\delta_r : R^n \to R^n,\ \delta_r(x) = rx, \text{ for } r \in R_+ = \{t : 0 < t < \infty\}.$$

Observe that the map, $\tau_x \circ g \circ \delta_r \circ \tau_{-y}$, which we consider below, is composed of a rotation and homothety with centre y and of the translation which takes y to x.

Suppose $0 < s < n$, $0 < t < n$ and $n < s+t$. Then the following results hold for any Borel (or even Suslin) sets $A, B \subset R^n$:

__Theorem__ 1. __For__ $0 \leq a < b < \infty$

$$t^{-2}(b^t - a^t)^2 C_s(A) C_t(B)$$

$$\leq c(n) \int_a^b r^{t-1} \iint C_{s+t-n}(A \cap (\tau_z \circ g \circ \delta_r)B)\, d\mathcal{L}^n z\, d\theta_n g\, d\mathcal{L}^1 r.$$

__Theorem__ 2. __Assume__ $C_t(B) > 0$. __There is__ $E \subset A$ __with__ $C_s(A \setminus E) = 0$ __and with the following property__: __For every__ $x \in E$ __there is__ $B_x \subset B$ __such that__ $C_t(B_x) > 0$ __and for all__ $y \in B_x$

$$C_{s+t-n}(A \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})B) > 0$$

__for__ $\theta_n \times \mathcal{L}^1$ __almost all__ $(g,r) \in O(n) \times R_+$.

Theorem 3. *If* $0 < H^s(A) < \infty$ *and* $0 < H^t(B) < \infty$, *then*

$$\dim A \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y}) B \geq s + t - n$$

for $H^s \times H^t \times \theta_n \times L^1$ *almost all* $(x,y,g,r) \in A \times B \times O(n) \times R_+$. *Here equality holds provided*

$$\liminf_{\rho \downarrow 0} \rho^{-t} H^t(B \cap B(y,\rho)) > 0 \quad \text{for} \quad y \in B.$$

In Theorem 3 the equality need not always be valid, but the above lower density condition holds for many interesting sets, for example for ordinary Cantor sets,more generally for all self similar sets, see [1].

References

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