Real Analysis Exchange Vol. 8 (1982-83)
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Some Remarks on σ-Porous Sets and Unilateral Derivates

The work outlined here represents joint work with Mike Evans, Krishna Garg, Ted Vessey, and I even did a bit myself. I'll motivate this area of effort with a problem presented by Krishna Garg at the 1976 Real Analysis Conference held at Syracuse, New York. The problem is simply to characterize the following set.

$$U(f) = \{x: D^+f(x) \neq D^-f(x) \text{ or } D_+f(x) \neq D_-f(x)\}$$

This is, of course, the set of points where unilateral derivates differ, and the situation which was known at the time was:

$f \varepsilon BV \Rightarrow U(f)$ is $G_{\delta\sigma}$, first category, measure zero.

$f \varepsilon C \Rightarrow U(f) = U_1 \cup U_2$ where U_1 is as above, and U_2 is an arbitrary G_σ.

Since 1976 a bit more has been determined and the situation now is:

(1) $f \varepsilon L$ (Lipschitz) $\Rightarrow U(f)$ is $G_{\delta\sigma}$, σ-porous

(2) $f \varepsilon BV \Rightarrow U(f)$ is $G_{\delta\sigma}$, first category, measure zero, but not necessarily σ-porous.

I'd like to discuss the converse of each of (1) and (2), and I'll begin with (2), and in particular, the following useful lemma of Zahorski.

LEMMA Z. <u>To every linear G_δ set E which does not contain an interval there corresponds a decreasing sequence of open sets G_n with</u>

$E = \bigcup_{n=1}^{\infty} G_n$ *and which has the following property: If the functions* f_n *are such that* $f_n(x) < \rho(G_n, x)$ *and are everywhere differentiable, then* $f(x) = \sum_{n=1}^{\infty} f_n(x)$ *exists and is continuous for all* x, *and* f *is differentiable for* $x \notin E$. *Indeed,* $f'(x)=0$ *if* $x \notin G_1$ *and* $f'(x) = \sum_{n=1}^{m} f_n'(x)$ *if* $x \varepsilon G_m - G_{m-1}$.

Additionally, the following lemma is useful.

LEMMA *If* K *is a nowhere dense perfect subset of* (a,b) *and* $\delta > 0$, *then there is a differentiable function* $f: \mathbb{R} \to [0,\infty)$ *such that:*

1. $f(x) < \rho(x,(a,b))$ *for all* x.
2. $Vf < 4(b-a)$ [V=*total variation*]
3. *For each* $x \varepsilon K$ *there is a* $y \varepsilon (x, x+\delta)$ *such that* $[f(y)-f(x)]/[y-x] > 1$.

Using the Vitalli Covering Theorem we can then prove:

THEOREM *Let* E *be a nowhere dense, measure zero,* G_δ *set, and let* $\varepsilon > 0$. *Then there is a bounded variation function* f *such that* f *is differentiable on* $\mathbb{R}-E$, $U(f)=E$, *and* $Vf < \varepsilon$.

One would think that this would be enough to handle the BV case, but the obvious things to do next don't work smoothly and as a consequence, the above theorem is the best we have. These results depend quite heavilly on the Vitalli Covering Theorem, and as such, it was our thought that such a theorem for σ-porous sets would be very handy in (1). A preliminary investigation into this is carried on within the paper entitled "Another note on σ-porous sets" which is in the Inroads section of this *Exchange*.