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Miklós Laczkovich, Department I. of Analysis,
Eötvös Loránd University, Budapest, Muzeum krt. 6-8.
Hungary H-1088

DIFFERENTIABLE RESTRICTIONS OF CONTINUOUS FUNCTIONS*

The following theorem was proved by Bruckner, Ceder and Weiss in [1].

<u>Theorem A.</u> For every continuous function f defined on a perfect set $P \subset \mathbb{R}$ there exists a perfect subset $Q \subset P$ such that the derivative of the restriction $f|_Q$ exists at each point of Q.

(Infinite derivatives are allowed and cannot be excluded. In fact, it is possible that $f'(x) = +\infty$ holds at every $x \in P$.)

If f is defined on an interval, then a stronger assertion can be proved.

<u>Theorem B.</u> Let f be continuous on the interval $[a,b]$. Then either

(i) there is a perfect subset $Q \subset [a,b]$ such that $f|_Q$ is constant,

or (ii) there is a perfect subset $Q \subset [a,b]$ such that $f'(x)$ exists at each point of Q.

Indeed, if (i) does not hold, then f fulfils condition (T_2) on $[a,b]$. Hence, by a theorem of Banach ([3], p.280), $f'(x)$ exists at the points of a non-countable

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* The work presented here will appear in [2].

set and this easily implies (ii).

The assertion of Theorem B is not true for continuous functions defined on perfect sets. It was shown in [1] that there exists a continuous function defined on a perfect set P such that f is strictly increasing and nowhere differentiable in P. However, our next result shows that if f is continuous on a set of positive measure, then the assertion of Theorem B "almost holds true".

Theorem 1. Let $P \subset \mathbb{R}$ be a perfect set of positive measure and let $f : P \to \mathbb{R}$ be continuous. Then one of the following assertions is true.

(*) There is a perfect subset $Q \subset P$ such that $(f|_Q)'(x) = 0$ for every $x \in Q$.

(**) f is differentiable at almost every point of a set $U \subset P$ which is everywhere dense and open relative to P and, for every $\varepsilon > 0$, there exists a perfect subset $Q \subset P$ such that $\lambda(P-Q) < \varepsilon$ and $f|_Q$ is differentiable at each point of Q.

(Observe that (*) is weaker than (i) and (**) is stronger than (ii).)

Theorem 1 obviously implies the following sharper form of Theorem A.

Corollary 2. Let $P \subset \mathbb{R}$ be perfect, $\lambda(P) > 0$ and let $f : P \to \mathbb{R}$ be continuous. Then there is a perfect subset $Q \subset P$ such that $f|_Q$ is differentiable (with finite derivative) at each point of Q .

Moreover, we can easily prove the following, apparently much stronger

<u>Corollary 3.</u> Let $P \subset \mathbb{R}$ be perfect, $\lambda(P) > 0$ and let $f : P \to \mathbb{R}$ be continuous. Then either

(a) there is a perfect subset $Q \subset P$ such that $f|_Q$ is infinitely differentiable on Q and, in addition,

$$(f|_Q)^{(n)}(x) = 0 \qquad (x \in Q)$$

holds for n large enough,

or (b) for every $\varepsilon > 0$ there is a perfect subset $Q \subset P$ such that $\lambda(P-Q) < \varepsilon$ and $f|_Q$ is infinitely differentiable on Q .

<u>References</u>

[1] A.M. Bruckner, J.G. Ceder and M.L. Weiss, On the differentiability structure of real functions, Trans. Amer. Math. Soc. <u>142</u> (1969), 1-13.

[2] M. Laczkovich, Differentiable restrictions of continuous functions, Acta Math. Acad. Sci. Hung., to appear

[3] S. Saks, Theory of the Integral (Hafner, New York, 1937)