Real Analysis Exchange Vol. 6 (1980-81)

H. W. Pu, Department of Mathematics, Texas A&M University,
College Station, Texas 77843

On the Derivative of a Nondecreasing Saltus Function

A nondecreasing saltus function is defined to be a function of the form

$$f(x) = \sum_{n=1}^{\infty} \varphi_n(x) , \quad \text{where} \quad \varphi_n(x) = \begin{cases} 0 & \text{for } x < a_n \\ b_n & \text{for } x = a_n \\ b_n + c_n & \text{for } x > a_n , \end{cases}$$

a_n is an arbitrary sequence of points, b_n and c_n are non-negative such that $b_n + c_n > 0$ and $\Sigma (b_n + c_n)$ converges.

In [2], Piranian solved Zahorski's seventh problem given in [6] by proving the following.

If A is countable and $A \in G_\delta$, then there is a discontinuous function f such that

$$f'(x) = \infty \quad \text{for } x \in A$$
$$= 0 \quad \text{for } x \notin A .$$

The function constructed there is in fact a nondecreasing saltus function.

A function that is a nondecreasing saltus function having derivative 0 or ∞ at each point is said to be of type (*) in

[4] and also in this note. Let $D(f)$ denote the set of points of discontinuity and $\Delta_\infty(f) = \{x : f'(x) = \infty\}$. It is clear, for a function f of type (*), that $D(f)$ is countable, $D(f) \subset \Delta_\infty(f)$ and the Lebesgue measure $|\Delta_\infty(f)| = 0$. Also, from known results ([5], [3]), we can get $\Delta_\infty(f) \in F_\sigma \cap G_\delta$. In [4], the authors study the existence of a function f of type (*) for given sets with the above properties to be $D(f)$ and $\Delta_\infty(f)$. Two theorems are obtained.

Theorem 1. If $E \in F_\sigma \cap G_\delta$ and $|E| = 0$, then there is a function f of type (*) such that $\Delta_\infty(f) = E$.

This theorem is proved by modifying Lipinski's work [1], Piranian's technique in [2] is also used.

Theorem 2. Let A be a countable set, $E \in F_\sigma \cap G_\delta$, $|E| = 0$ and $A \subset E$. Then there exists a function f of type (*) such that $D(f) = A$ and $\Delta_\infty(f) = E$ if and only if $E \subset \bar{A}$ and $E \cap I(\bar{A}) \subset A$, where $\bar{A}$ is the closure of A and $I(\bar{A})$ is the set of $x \in \bar{A}$ such that $(x - \delta, x) \cap \bar{A} = \emptyset$ or $(x, x + \delta) \cap \bar{A} = \emptyset$ for some $\delta > 0$.

The proof is based on two lemmas:

Lemma 1. For a countable set A, there is a function f of type (*) such that $D(f) = A$ and $\Delta_\infty(f) = \bar{A}$ if and only if $|\bar{A}| = 0$ and $A \supset I(\bar{A})$.

Lemma 2. Let A be countable, $A \subset E \subset \bar{A}$, $E \cap I(\bar{A}) \subset A$ and $|E| = 0$. If F is any closed subset of E, then there is a closed set K such that $F \subset K \subset E$, $K = \overline{K \cap A}$ and $I(K) \subset K \cap A$. Moreover, K can be chosen bounded if F is bounded.

References

[1] J. S. Lipinski, Sur la dérivée d'une fonction de sauts, Colloq. Math., 4 (1957), 195 - 205.

[2] G. Piranian, The derivative of a monotonic discontinuous function, Proc. Amer. Math. Soc., 16 (1965), 243 - 244.

[3] D. Preiss, Approximate derivatives and Baire classes, Czech. Math. J., 21 (1971), 373 - 382.

[4] H. W. Pu and H. H. Pu, The derivative of a nondecreasing saltus function, (to be submitted).

[5] W. H. Young, On the infinite derivates of a function of a single real variable, Arkiv för Math. Astr. och Fysik, 1 (1903), 201 - 204.

[6] Z. Zahorski, Sur la premiere dérivée, Trans. Amer. Math. Soc., 69 (1950), 1 - 54.

Received April 29, 1980