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GENERALIZED DERIVATIVES

The classical assertion that there exist continuous, nowhere differentiable functions can be generalized in various ways. One such possibility was shown by L. Filipczak in [1]. He constructed a periodic continuous function whose upper and lower symmetric derivates are ∞ and $-\infty$, respectively, at each point. I would like to mention some theorems of J.C. Georgiou and myself that together generalize Filipczak's result.

Let r be a natural number and let $a_0 < a_1 < \ldots < a_r$. There are b_j such that $\sum_{j=0}^{r} b_j a_j^k = 0$ for $k = 0, 1, \ldots, r-1$ and $\sum_{j=0}^{r} b_j a_j^r = r!$. For each finite real function f on $R = (-\infty, \infty)$ and each pair of real numbers x, h with $h \neq 0$ we define $L(f,x,h) = \sum_{j=0}^{r} b_j f(x + a_j h)$, $\lambda(f,x,h) = h^{-r} \cdot L(f,x,h)$. It is easy to see that $\lambda(f,x,h) \to f_{(r)}(x)$ $(h \to 0)$, if the r-th Peano derivative $f_{(r)}(x)$ exists. If $a_j = j - \frac{r}{2}$ for $j = 0, \ldots, r$, then $\lim \lambda(f,x,h)$ means the r-th Riemann derivative of f at x.

Now we may ask whether there is an f with the following property:

(P) The function f has a continuous derivative of order $r-1$ on R and, for each $x \in R$,

$$\limsup_{h\uparrow 0} \lambda(f,x,h) = \limsup_{h\downarrow 0} \lambda(f,x,h) = \infty ,$$

$$\liminf_{h\uparrow 0} \lambda(f,x,h) = \liminf_{h\downarrow 0} \lambda(f,x,h) = -\infty .$$

The following assertion is helpful:

(A) Let F be a continuous, periodic function on R such that

(Q) for each $x \in R$ there are $h_1, h_2 \in (-\infty,0)$ and $h_3, h_4 \in (0,\infty)$ with

$$(-1)^i \cdot L(f,x,h_i) > 0 \quad (i = 1,2,3,4) .$$

Then there is an f with property (P).

It is possible to indicate the proof of (A) as follows: We approximate F by a periodic function G with a continuous derivative of order r, choose a large natural number a, define a suitable positive number b (we need, in particular, $a^{r-1}b < 1 < a^r b$) and set $f(x) = \sum_{k=0}^{\infty} b^k G(a^k x)$ for each x.

It can be proved that under the assumption $a_0 \ldots a_r \neq 0$ (this is obviously fulfilled, if r is odd and $a_j = j - \frac{r}{2}$) either $F(x) = \cos x$ or $F(x) = \cos x + \sin 2x$ has property (Q). Taking $r = 1$, $a_0 = -1$, $a_1 = 1$ and applying (A) we obtain Filipczak's result.

If $a_0 \dots a_r = 0$, then the situation is not so simple. If $r = 2$ and $a_1 = 0$, then there is no f with property (P) and, consequently, no F with property (Q). We have been able to find an F with property (Q) in the following cases: $3 \leq r \leq 12$ and $a_j = j - \frac{r}{2}$; $r = 2$ and $a_0 a_2 = 0$; $r = 3$ and $a_0 a_3 = 0$. However, we have not been able to find an $r > 2$ and $a_0, \dots, a_r$ for which such an F does not exist.

On the other hand, by means of an assertion analogous to (A) we proved that, in any case, there is a function f with a continuous derivative of order $r - 1$ such that $\limsup_{h \uparrow 0} |\lambda(f,x,h)| = \limsup_{h \downarrow 0} |\lambda(f,x,h)| = \infty$ for each $x \in R$.

Reference

[1] L. Filipczak, Exemple d'une fonction continue privée de dérivée symétrique partout, Coll. Math. XX (1969), 249-253.

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