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Having Small Selectors

A linear set S will be called a selector with respect to translations for a linear E if every translate E + x has non-empty intersection with S. Equivalent forms of that relationship between S and E are: the set of distances between x and y, where $x\varepsilon E$ and $y\varepsilon S$ forms the set of all positive numbers; or: the product $E \times S$ when projected in $\mathbb{R}^2$ -plane along one of the two diagonals on one of the coordinate-axis, covers the whole axis (or, using a more picturesque form of speech, $E \times S$ is "opaque" to a bunch of parallel rays running along one of the diagonals).

My talk is about having small selectors. Let me give the general idea what it is about. Suppose that there is given some class $\mathbb{E}$ of subsets of $\mathbb{R}^1$ and we wish for any set from that class to have a selector with respect to translations, which would have certain desirable properties. For instance, we may wish that selector to be "small" in a certain sense which is intuitive and definable in terms of structural properties of $\mathbb{R}^1$ (for instance, in topological terms or in terms of measure). The question is: could we have for each set from $\mathbb{E}$ a selector which would be "small"? That question becomes interesting, when the class $\mathbb{E}$ is sufficiently large but is not restricted just to "large" sets (again, "largeness" must be understood in an appropriate sense). To be asking such and similar questions and answering them if possible, is the guiding idea of these investigations, that I am going to talk about.

As a very natural means of "smallness" in topological terms, take the nowhere-density property. As $\mathbb{E}$ take the first trial, the class of all the uncountable subsets of $\mathbb{R}^1$. As a first-trial conjecture ask the question: is there for each set from $\mathbb{E}$ a nowhere-dense selector with respect to translations? That question is easily answered in the negative: using the standard technique of well-ordering as well as the Continuum Hypothesis, an uncountable E is defined whose appropriate translate is disjoint with every nowhere-dense perfect subset of $\mathbb{R}^1$.

It is clear, that if we want to obtain a result of the described kind, the class of sets must be somehow restricted. Let us try to make that restriction as weak as possible, using some possibly natural restricting condition.

As R.O. Davies has noted (in a letter to me), the two known results: (a) an uncountable analytic set contains a perfect subset (Souslin, Alexandrov), and (b) a set with Borel (C) property (that is a set which can be covered by a sequence of open intervals of lengths prescribed in advance) cannot contain a perfect subset (Spilrajn), taken combinely imply that an analytic set cannot have (C) property. That in turn implies (Davies, ibid.) easily for such sets the existence of nowhere-dense selectors with respect to translations. Indeed, for a set E not having (C) property means, that a certain sequence of open intervals, no matter in what positions, is unable to cover E, and certainly, none of its translates, thus the complement of the union of those intervals is a selector. Its nowhere-density can be secured by the choice of centers of the intervals at points forming a dense subset of $\mathbb{R}^1$.

The above points out to the selection of the class of uncountable analytic sets as an adequate class for our considerations. Henceforth $\mathbb{E}$ will be assumed to be that class.

Other classes of transformations may be used in place of translations and the same questions asked. Among the variety of possible classes let us just mention two, as appearing to generate natural questions of particular interest: the class Lip (α,β) of two-way-Lipschitzian distortions, where α and β, $0<\alpha\leq 1\leq\beta$ are the lower and upper bounds of distortion, and the narrower class consisting of combinations of translations and linear dilations. Note, that the lower bound must be positive, should we expect to have a nowhere-dense selector. In terms of the product EXS, the E and S relationship takes now the form: EXS is "opaque" for any curved "ray" running in the $\mathbb{R}^2$ -plane, whose tangents differ not too much in direction from one of the two diagonals.

The above presented result about the existence of nowhere-dense selectors with respect to translations carries over on the case when translations are replaced by Lip (α,β). It is not a very deep result. We enter a new area of difficulties when in addition to nowhere-density we demand the selector to be small in terms of measure. R. O. Davies has proposed recently an elegant proof (in a letter to me) showing that in the case of translations (which is to say, for Lip (1,1)) a measure zero for the selector may be had. How it is for $\alpha<\beta$, or even for the narrower class consisting of translations with dilations, is not clear as yet. It seems doubtful that the same result could be extended on any of those classes, but nothing has been proved or disproved for the time being.

When a set S is anything but of measure zero, there is a difficulty in choosing an adequate numerical characteristic measuring its "smallness" in terms of measure. For instance, the average density in an interval J (here J is a bounded interval and $|\cdot|$ stands for measure) $|S \cap J|/|J|$ is not good for that, yielding values close to 1 for sets of positive, however small, measure, at the same time being close to 0 for some sets of infinite measure. For that reason I proposed to modify the question itself, directing it at the characterization of a family rather than of an individual selector. Let Ω be a space with convergence, $S(\omega)$, $\omega \in \Omega$ a selector-valued function defined over Ω, ω_0 a selected point in Ω. The upper-limit-average density at ω_0 $S(.)$ on J we define as

$$\text{ULAD} = \text{ULAD}\, [\, S(.), \omega_0,\ J] = \overline{\lim_{\omega \to \omega_0}}\, |S(\omega)\ J|/|J|$$

If it turns out that the above expression is uniformly over the choice of J bounded-away from 1, or even equal 0, it would mean "smallness in limit" of $S(\omega)$, regardless of the "cluster-structure" of individual sets S.

Now, let $0<\alpha\leq 1\leq \beta$ and define for an E from IE

$$\lambda_{\alpha\beta}\ (E) = \inf\, [\, \sup_{|J|<\infty} \text{ULAD} :\ S(.),\ \Omega,\ \omega_0]$$

where S(.) is let to vary over all the selector-valued with respect to Lip (α,β) functions and Ω and ω_0 over all possible choices of a suitable space and an element in it. The smallness of $\lambda_{\alpha\beta}(E)$ is then indicative of how well we can do in having the respective selectors small "in limit" in terms of measure. Obviously, in the case when there is a selector of measure zero, we have $\lambda_{\alpha\beta}(E) = 0$.

$\mathbb{R}^n$ instead of $\mathbb{R}^1$ may be made the setting for our quest, using analogous classes of transformations. This time the condition $\chi=\beta=1$ turns the Lip $(|,|)$ into the class of isometries, a class certainly deserving special consideration. It should be noted, that results obtained for $\mathbb{R}^1$ for translations, generate trivially certain results in $\mathbb{R}^n$ for the class of translations there. For instance, any set from E projects on some coordinate-axis onto a linear set having a perfect subset, a fact implying the existence in $\mathbb{R}^n$ (by taking a cylindrical set) of a (n-1)-dimensional selector with respect to translations. No results as to the possibility to lower this estimate for dimension are known to us. The above mentioned proof of existence of selectors with respect to translaitons of measure zero, proposed by R. O. Davies and drafted for $\mathbb{R}^1$, may be adjusted so that it is good for $\mathbb{R}^n$ and for the class of isometries as well. There is a possibility open to push the inquiry even further, asking whether that result may be improved by having selectors of even lower-dimensional measure zero (or small in terms of Hausdorff dimension). Nothing in that direction is known to us.

Define Δ_n,α,β for $0<\alpha\lesssim 1\leq\beta$ as follows: from each of the balls of diameter $\alpha+\beta$ which form a dense packing of $\mathbb{R}^n$, remove a concentric open ball of diameter α (Note: the mutual distances between the removed balls are not smaller than β).

Let the resulting set in $\mathbb{R}^n$ be Σ. Let $\Sigma\omega$ be a homotetical dilation (contraction) of Σ with coefficient ω, $\omega>0$. $\Sigma\omega$ as a set-valued function of ω limit-average-density as $\omega\to 0$, which is independent on J (understood here as interval in $\mathbb{R}^n$). $\Delta n,\alpha,\beta$ is that density. Defining now in $\mathbb{R}^n$ the index $\lambda_{n,\alpha,\beta}(E)$ analogously to what we have done for

$\mathbb{R}^1$, we obtain for it an estimate:

$$\lambda_{n,\alpha,\beta}(E) \leq [\Delta_{n,\alpha,\beta}]^{-1} - 2$$

It is easily seen that $\Delta 1,1,1 = \frac{1}{2}$, yielding $\lambda 1,1 = 0$ in one-dimensional case (and only then). Since, as have been mentioned, when $\alpha=\beta=1$ we have simply a selector of measure zero, we have a result which supercedes in that case the need for the above estimate. However, for $\alpha<\beta$ we have nothing as good so far, thus the estimate is the best that there is known in that respect at the moment. Our joint work with R. O. Davies is still in progress and it remains to be seen which final form the result will take.

In closing, let me mention another line of inquiry yet, which has been only superficially explored so far, and which consists in putting restrictions upon the structure of the sets of $\mathbb{E}$, thus restricting $\mathbb{E}$ to its subclasses. As could be expected, stronger degrees of smallness may be then obtained for the selectors.