Real Analysis Exchange Vol. 5 (1979-80)
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DISTRIBUTIONAL DERIVATIVES AND ABEL SUMMABILITY OF ULTRASPHERICAL EXPANSIONS

Introduction:

Let $C_n^\mu(x)$ denote the ultraspherical (Gegenbauer) polynomial of degree n and $\{a_n\}_{n=0}^\infty$ be a sequence of complex numbers such that $\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} < 1$. The following facts can be easily shown (see [1], [4]):

i) The set $\{C_n^\mu(x)\}_{n=0}^\infty$ is orthogonal and complete over $(-1,1)$ with respect to the measure $(1-x^2)^{\mu-\frac{1}{2}}dx$, with $\int_{-1}^{1}(1-x^2)^{\mu-\frac{1}{2}}C_n^\mu(x)C_m^\mu(x)dx = h_n^\mu\delta_{mn}$

where $h_n^\mu = \frac{2^{1-2\mu}\,\pi\Gamma(n+2\mu)}{n!\,(\mu+n)\,[\Gamma(\mu)]^2}$.

ii) $C_n^\mu(x)$ satisfies the ordinary differential equation

$(1-x^2)\,y''-(2\mu+1)xy' + n\,(n+2\mu)y = 0.$

iii) The function $f(x,y) = \sum_{n=0}^{\infty} a_n\,(x^2+y^2)^{\frac{n}{2}}\,C_n^\mu\left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right)$

is a solution for the singular partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{2\mu}{y}\,\frac{\partial f}{\partial y} = 0.$$

It is also well known that if $\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} < 1$, then the series $f(z) = \sum_{n=0}^{\infty} a_n C_n^{\mu}(z)$ converges to a holomorphic function in some neighborhood of $[-1,1]$. But if $\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} = 1$, then the series may diverge everywhere (in the classical sense). However, we have shown in a recent work [6] that the series converges to a hyperfunction on $[-1,1]$. A hyperfunction on $[-1,1]$ is a continuous linear functional on the space of analytic functions on $[-1,1]$ provided with a certain topology [2]. If the growth rate of the sequence $\{a_n\}_{n=0}^{\infty}$ is restricted, e.g. $a_n = O(n^p)$ for some integer p, then we show that the series converges to a generalized function (Schwartz distribution) on $(-1,1)$ which is a continuous linear functional on the space of C^{∞}-functions with support in $(-1,1)$. Since generalized functions and continuous functions are closely related e.g. every generalized function f with compact support is the kth distributional derivative of some continuous function $F(x)$, we will be able to study the behavior of the series $f(x) = \sum_{n=0}^{\infty} a_n C_n^{\mu}(x)$ via $F(x)$. Instead of looking at the global properties of $f(x)$ as it is usually done we shall examine the local behavior of $F(x)$ in some neighborhood of $x_0 \in (-1,1)$ and try to interpret it in terms of $f(x)$. To be more specific, we shall show that if the normalized kth Peano derivative of

$F(x)$ at x_0 exists, and is equal to say γ, then the series $\sum_{n=0}^{\infty} a_n C_n^{\mu}(x_0)$ is Abel summable to γ. Under a slightly stronger condition it can be shown that the associated power series $\phi(z)= \sum_{n=0}^{\infty} a_n h_n^{\mu} z^n$, which converges for $|z| < 1$, approaches its boundary value $\phi(\beta)$ as $z \to \beta$ radially where $|\beta| = 1$ and $x_0 = \frac{1}{2}(\beta+\frac{1}{\beta})$.

2. <u>Definitions and Notations</u>:

Let I denote the interval $(-1,1)$ and $C_0^{\infty}(I)$ denote the space of C^{∞}-functions with support in I. A generalized function (g.f) f on I is a continuous linear functional on the topological linear space $C_0^{\infty}(I)$. The action of f on $\phi(x)\epsilon C_0^{\infty}(I)$ is denoted by $\langle f(x),\phi(x)\rangle$. The g.f $f(\lambda x + x_0)$ is defined by $\langle f(\lambda x + x_0), \phi(x)\rangle = \langle f(x),\frac{1}{\lambda}\phi(\frac{x-x_0}{\lambda})\rangle$

We say that $f(x)$ has a value at x_0 if $\lim_{\lambda \to o}\langle f(\lambda x+x_0),\phi(x)\rangle$ exists for all $\phi \epsilon C_0^{\infty}(I)$. It has been shown [3] that f has the value λ at x_0 if and only if there exists an integer $k \geq 0$ and a continuous function $F(x)$ such that $F^{(k)} = f$ and

$\lim_{x\to x_0} \frac{F(x)}{(x-x_0)^k} = \frac{\gamma}{k!}$. Clearly, this is equivalent to saying that the normalized <u>kth</u> Peano derivative of $F(x)$ at x_0 exists and is equal to γ.

3. Summability theorems:

The results of this section along with the details of the proofs will be published elsewhere.

Proposition Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that $a_n=O(n^p)$ for some integer p. Then there exists a generalized function f such that the series $\sum_{n=0}^{\infty} a_n C_n^{\mu}$ converges in I to f.

Proof: Consider the function $F(x) =$

$$\sum_{n=0}^{\infty} \frac{(-1)^k a_n}{(n+\mu)^{2k}} C_n^{\mu}(x) \chi_I \qquad - (1)$$

where $2k \geq 2\mu + p + 1$. $F(x)$ is continuous on $[-1,1]$ since $\max_{x\in[-1,1]} |C_n^{\mu}(x)| = n^{2\mu-1}$. Using the facts that $L\, C_n^{\mu}(x) = -(n+\mu)^2 C_n^{\mu}(x)$ where $L= (1-x^2)\frac{d^2}{dx^2} - (2\mu+1)x\frac{d}{dx} - \mu^2$ and that L is a continuous operator on the space of generalized functions we can apply L to eq.(1) and this finishes the proof.

Theorem 1.

Let f be a generalized function with support in I given by $f(x) = \sum_{n=0}^{\infty} a_n C_n^{\mu}(x)$. If f has a value γ at $x_0 \in I$, then $\sum_{n=0}^{\infty} a_n C_n^{\mu}(x_0)$ is Abel summable to γ.

Sketch of the proof: By the hypothesis there exist a non-negative integer k and a continuous function F(x) such that $F^k(x) = f(x)$, and

$$\lim_{x\to x_0} \frac{F(x)}{(x-x_0)^k} = \frac{\gamma}{k!} .$$ Therefore,

$$\sum_{n=0}^{\infty} a_n C_n^{\mu}(x_0) r^n = \sum_{n=0}^{\infty} h_n^{-\mu} C_n^{\mu}(x_0) r^n \left\langle f, (1-x^2)^{\mu-\frac{1}{2}} C_n^{\mu}(x) \right\rangle$$

$$= \sum_{n=0}^{\infty} (-1)^k h_n^{-\mu} C_n^{\mu}(x_0) r^n \left\langle F, \frac{d^k}{dx^k}(1-x^2)^{\mu-\frac{1}{2}} C_n^{\mu}(x) \right\rangle$$

$$= \int_{-1}^{1} \left[\frac{F(x)}{(x-x_0)^k} k!\right] G^k(x_0,x,r)\,dx \qquad \text{where}$$

$$G^k(x_0,x,r) = \frac{(x_0-x)^k}{k!} \sum_{n=0}^{\infty} r^n h_n^{-\mu} C_n^{\mu}(x_0) \left(\frac{d^k}{dx^k}(1-x^2)^{\mu-\frac{1}{2}} C_n^{\mu}(x)\right) \qquad (2)$$

We show that $G^k(x_0,x,r)$ is a quasi-positive kernel and then we use the theory of singular integrals [5] to show that the limit of eq.(2) when $r \to 1^-$ is γ.

Theorem 2

Let f be given as in Theorem 1. Suppose that

the $[\mu]$-th distributional derivative of f has a value

γ at $x_0 \in I$.

Then $\phi(z) = \sum_{n=0}^{\infty} a_n h_n^{\mu} z^n \to \phi(\beta)$ as $z \to \beta$ radially where

$x_0 = \frac{1}{2}(\beta + \frac{1}{\beta})$

Sketch of the proof: We show that $\phi(z)$ can be given by

$$\phi(z) = \left\langle f, \frac{(1-x^2)^{\mu-\frac{1}{2}}}{(1-2xz+z^2)^{\mu}} \right\rangle$$

$$\doteq (-1)^k \left\langle F, \frac{d^k}{dx^k} \frac{(1-x^2)^{\mu-\frac{1}{2}}}{(1-2xz+z^2)^{\mu}} \right\rangle .$$

Using an argument similar to the one given in Theorem 1 yields the result.

References:

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