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Generalized bounded variation - recent results and open questions.

A result of Goffman and Waterman characterizes the class of regulated functions (GW) which, for every change of variable, have an everywhere convergent Fourier series. Baernstein and Waterman have characterized the class of continuous functions (UGW) which, for every change of variable, have uniformly convergent Fourier series. Both characterizations involve sums of the form $\sum_1^N f(I_n)/n$ where $I_n = [a_n, b_n]$, $f(I_n) = f(b_n) - f(a_n)$, and the intervals I_n are ordered, i.e., either $a_n < b_n \le a_{n+1}$ for every n or $b_{n+1} \le a_n < b_n$ for every n.

The definition of HBV is that sums of the form $\sum_1^N |f(I_n)|/n$ are uniformly bounded, but here the I_n are only nonoverlapping, the requirement that they be ordered is omitted. We know that $f \in HBV$ can replace the hypothesis $f \in BV$ in the Dirichlet-Jordan theorem. The definitions of GW and UGW suggest that the notion of order be added to create another class of functions whose relation to the previous classes can then be considered.

If we add the requirement that $\{I_n\}$ be ordered we obtain a class of functions of generalized bounded variation which we call OHBV. It is clear that OHBV $\supset$ HBV. With suitable norms both classes are Banach spaces. Belna has just constructed a continuous function $f \in OHBV$ such that $f \notin HBV$. Using his example we can show that HBV is of first category in OHBV. It may also be shown that $f \notin GW$. If we define $V_{OH}(g,x) = \sup\{\Sigma|g(I_n)|/n: \{I_n\} \subset [a,x]\}$, the sequences $\{I_n\}$ ordered, it may be shown that $V_{OH}(g,x)$ may have a jump at a point of continuity of $g \in OHBV$, unlike the harmonic variation.

Summarizing the inclusion relations between the various classes of functions we have

$$HBV_C \subset UGW \begin{cases} \subsetneq OHBV_C \\ \subset GW_C \end{cases}$$

and

$$OHBV_C - GW \neq \emptyset$$

Here X_C denotes the continuous functions in class X. The most interesting questions here are

1. Is $HBV_C = UGW$?
2. Is $HBV = GW$?

Another question is

3. Is $GW - OHBV = \emptyset$?

The method of Belna is applicable only to OHBV and not to $O\Lambda BV$ defined in the obvious way. Thus we ask

4. Is $O\Lambda BV \neq \Lambda BV$ for $\Lambda \neq H$?

Turning to other types of variation, let Φ and Ψ be complementary functions in the sense of W. H. Young's inequality. $f \in \Phi\text{-}BV$ if there is a $C < \infty$ such that for some $k > 0$, $\Sigma\Phi(k|f(I_n)|) < C$ for every partition $\{I_n\}$ of the domain of f. We know that $\Phi\text{-}BV \subset HBV$ if $\Sigma\Psi(1/n) < \infty$. A natural question is

5. Is HBV = $\bigcup\Phi$-BV, the union being extended over all Φ with $\Sigma\Psi(1/n) < \infty$?

If f is a regulated function, we may "complete" its graph by adjoining to it each line segment $[(x,\underline{f}(x)),(x,\bar{f}(x))]$, where $\underline{f}(x) = \min\{f(x), \liminf_{t\to x} f(t)\}$ and $\bar{f}(x)$ is analogously defined. Let $n(y)$ be the natural extension of the notion of Banach indicatrix that is obtained by considering the intersection of horizontal lines with the completed graph. We say that a function is in the Garsia-Sawyer class (GS) if

$$\int \log^{+} n(y)\,dy < \infty \quad .$$

We can show that $GS \subset HBV$. However GS is not a linear space and so $GS \neq HBV$. A natural question is

6. If $\overline{GS}$ is the closed linear space of GS in HBV, is $\overline{GS} = HBV$?