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Changes of Variable which Preserve Almost Everywhere Approximate Differentiability

The purpose of the above paper is to characterize those inner homeomorphisms which preserve approximate derivability a.e. Consider the following classes of homeomorphisms H on a closed interval:

$A = \{H \mid F'ap\,(x)$ exists a.e. implies $(F \circ H)'ap(x)$ exists a.e.$\}$

$B = \{H \mid F'(x)$ exists a.e. implies $(F \circ H)'(x)$ exists a.e.$\}$

$C = \{H \mid H^{-1}(x)$ is absolutely continuous$\}$

$D = \{H \mid H$ is continuous a.e. in the density topology$\}$.

In D, continuity in the density topology means that the density topology is considered to be the topology of both the range and domain; continuity at a point refers to the usual topological notion. The sole purpose of the above paper is to prove that all four of these sets are the same and that it is no further restriction to assume that F is continuous. This is done by showing that $D \subset C$ and $C \subset D$, $C \cap D \subset A \cap B$ and showing that given H with H^{-1} not absolutely continuous, there is a continuous G which is differentiable almost everywhere such that $G \circ H$ is not approximately derivable on a set of positive measure.

That $D \subset C$ is due to the fact that if H^{-1} is not absolutely continuous then H takes a set of positive measure onto a set of measure 0 . Then at each point of density of the set of positive measure, H is easily seen to be discontinuous in the density topology.
That $C \subset D$ follows from an application of Lusin's theorem to the measurable function $H'(x)$, the fact that for $H \in C$, $\{x | H'(x) = 0$ or $\pm \infty$ or does not exist$\}$ is of measure 0 , and that no $H \in C$ can take a set of measure 0 into a set of positive measure. The conclusion is that each such H is continuous at any point x where the derivative of H exists, is non-zero, and $H'(x)$ is approximately continuous.

That $C \cap D \subset A \cap B$ is a consequence of the formula:

$$\frac{F(H(x+h))-F(H(x))}{h} = \frac{F(H(x+h))-F(H(x))}{H(x+h)-H(x)} \qquad \frac{H(x+h)-H(x)}{h}$$

the fact that $H(x+h)-H(x)$ is never 0 , and the properties defining $H \in C$ and $H \in D$. The chain rule is thus shown to hold for the composition.

Finally, if H is a homeomorphism whose inverse is not absolutely continuous, H takes a perfect set of positive measure P onto a set Z of measure 0. If G is a nowhere approximately derivable function and G_0 equals G on P and is linear on intervals contiguous to P, then $F = G_0 \circ H^{-1}$ is easily seen to be differentiable a.e. (F is monotone on intervals contiguous to Z). However, $F \circ H$ is not approximately derivable a.e. (namely, at every point of density of P).

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