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SOME EXTENSIONS OF THE GAMMA AND BETA FUNCTIONS

The gamma function was discovered by Euler in 1729, and the first book on elliptic functions was published in 1829, so 1979 seemed an appropriate time to reconsider them by looking at an extension of the gamma function which is related to elliptic functions. The q-gamma function, $\Gamma_q(x)$, is a solution of

$$\Gamma_q(x+1) = \frac{(1-q^x)}{(1-q)}\,\Gamma_q(x) \,. \tag{1}$$

When $0 < q < 1$ there is a unique solution with $\Gamma_q(1) = 1$ and $\log \Gamma_q(x)$ convex for $x > 0$. It is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x} \,, \tag{2}$$

where $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$. When $x = n+1$ it reduces to $(n;q)! = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. This is an increasing function of q. For fixed $x > 0$, $\Gamma_q(x)$ increases in q when $0 < x < 1$ and $x > 2$ and decreases in q when $1 < x < 2$. From this and the Bohr-Mollerup theorem it is easy to see that $\lim_{q\to 1} \Gamma_q(x) = \Gamma(x)$, [1]. Formula (2) is analogous to Euler's infinite product definition of $\Gamma(x)$. It is natural to ask if $\Gamma_q(x)$ can be used to evaluate any integrals. There is one of Ramanujan which can be considered as a natural extension of an integral equivalent to the beta function.

$$(3)\qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{x-1}\,dt = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,ds\ .$$

From now on take $0 < q < 1$. Ramanujan evaluated

$$\int_0^\infty s^{x-1}\ \frac{(-as;q)_\infty}{(-s;q)_\infty}\,ds\ .$$

When $a = q^{x+y}$ this can be evaluated as

$$(4)\qquad \int_0^\infty s^{x-1}\,\frac{(-sq^{x+y};q)_\infty}{(-s;q)_\infty}\,ds = \frac{\Gamma(x)\Gamma(1-x)\Gamma_q(y)}{\Gamma_q(x+y)\Gamma_q(1-x)}\quad x,y > 0\ .$$

The two published proofs of this evaluation, both given by Hardy, use Cauchy's theorem to reduce the integral to a sum, which is evaluated by the q-binomial theorem

$$(5)\qquad \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n}\,x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}\ .$$

A simple proof of (4) was given, using functional equations and the special case $x+y = 1$. A similar proof can be used to evaluate another extension of (3) due to Ramanujan. This is

$$(6)\qquad \sum_{-\infty}^{\infty} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty}\,t^n = \frac{(at;q)_\infty(\frac{q}{at};q)_\infty(q;q)_\infty(\frac{b}{a};q)_\infty}{(t;q)_\infty(\frac{b}{at};q)_\infty(a;q)_\infty(\frac{q}{a};q)_\infty}\ .$$

When $b = aq$, this becomes

$$(7) \qquad \sum_{-\infty}^{\infty} \frac{t^n}{1-aq^n} = \frac{(at;q)_\infty \left(\frac{q}{at};q\right)_\infty (q;q)_\infty^2}{(t;q)_\infty \left(\frac{q}{t};q\right)_\infty (a;q)_\infty \left(\frac{q}{a};q\right)_\infty} .$$

The sum (6) is a discrete approximation to the last integral in (3), when $t = q^x$. Point masses are located at q^i ; $i = 0, \pm 1, \cdots$. The special case in (7) corresponds to the case $x+y = 1$, or $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$. The pair of infinite products on the right, $(at;q)_\infty (q/at;q)_\infty$ and $(t;q)_\infty (q/t;q)_\infty$ are theta functions, since the triple product identity of Jacobi is

$$(t;q)_\infty (q/t;q)_\infty (q;q)_\infty = \sum_{-\infty}^{\infty} (-1)^n q^{(n^2-n)/2} t^n .$$

The ratio of two theta functions is an elliptic function, and (7) contains as special cases Jacobi's Fourier series expansions of snx, cnx and dnx .

The special cases $y = \infty$ of (4) and $b = 0$ of (6) give measures that have moments of all orders. By an appropriate choice of the parameter a these moments are equal, so give an indeterminate Stieltjes moment problem. The polynomials orthogonal to these measures have been found. They generalize the Laguerre polynomials. D. Moak used them to find the extreme measures for which polynomials are complete in L^2 . I think this is the first moment problem for which the extreme measures have been found explicitly.

There are other integrals that can be evaluated using the q-gamma function. One is

$$\int_{-1}^{1} \prod_{n=0}^{\infty} \frac{(1-2xq^n+q^{2n})(1+2xq^n+q^{2n})}{(1-2axq^n+a^2q^{2n})(1+2bxq^n+b^2q^{2n})} \frac{dx}{\sqrt{1-x^2}} .$$

When $a = q^\alpha$, $b = q^\beta$ and $q \to 1$ this goes to

$$2^{\alpha+\beta} \int_{-1}^{1} (1-x)^{\alpha-\frac{1}{2}} (1+x)^{\beta-\frac{1}{2}} dx .$$

The special case $a = b = q^\alpha$ gives a measure that generalizes $(1-x^2)^{\alpha-\frac{1}{2}} dx$. The polynomials orthogonal with respect to this measure were discovered by L. J. Rogers and used by him to derive the Rogers-Ramanujan identities. Rogers found many important properties of these polynomials, but was unaware they were orthogonal. The orthogonality was found two years ago as a special case of a more general orthogonality found by Askey and Wilson [3]. Another proof of this orthogonality and derivation of some of Rogers' formulas, and some new ones, is given by Askey and Ismail [2].

[1] R. Askey, The q-gamma and q-beta functions, Applicable Analysis 8 (1978), 125-141.

[2] R. Askey and M. Ismail, A generalization of ultraspherical polynomials, to appear in memorial volume for P. Turán.

[3] R. Askey and J. Wilson, Some orthogonal polynomials that generalize the Jacobi polynomials, to appear.