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SOME NEW INEQUALITIES IN COMPLEX ANALYSIS (AND WHAT THEY DO)

The interplay between Brownian motion and analytic functions is a rich source of problems and results.

The following inequality was suggested and first proved in the context of Brownian motion.

Let F be analytic in the unit disc D and

$$N_\alpha(F)(\theta) = \sup |F(z)|,$$

where $0 < \alpha < 1$ and the supremum is taken over all z in the interior of the smallest convex set containing the circle $|z| = \alpha$ and the point $e^{i\theta}$ on the boundary of D. Let m denote Lebesgue measure on $[0,2\pi)$.

THEOREM 1. Suppose that F and G are analytic in D with $F(D) \cap G(D)$ nonempty and, for almost all θ, the nontangential limit $G(e^{i\theta})$ of G at $e^{i\theta}$ exists and satisfies $G(e^{i\theta}) \notin F(D)$. Then

$$m(N_\alpha(F) > \lambda) \leq c\, m(N_\alpha(G) > \lambda), \quad \lambda > 0, \tag{1}$$

where

$$c = c_\alpha \frac{1 + |a|}{1 - |a|} \frac{1 + |b|}{1 - |b|},$$

$a, b \in D$ and satisfy $F(a) = G(b)$, and the choice of the positive real number c_α depends only on α.

Many variations of this inequality hold, for example, if F and G are replaced in (1) by $\operatorname{Re} F$ and $\operatorname{Re} G$. However, if the nontangential maximal functions of F and G are replaced by their radial maximal functions, then (1) does not hold.

This theorem leads to new answers to the following question. If $S \subset \mathbb{C}$, under what conditions does

$$(2) \qquad G(e^{i\theta}) \in S \text{ a.e.} \Rightarrow G(D) \subset S?$$

For example, let $M^{\log}$ denote the class of all G analytic in D that satisfy

$$\liminf_{\lambda\to\infty} (\log \lambda)\, m(N_\alpha(G) > \lambda) = 0.$$

The uniform Nevanlinna class N^+ is a proper subclass of $M^{\log}$.

THEOREM 2. If S is compact with a connected complement, then (2) holds for all $G \in M^{\log}$.

In particular, if $G \in M^{\log}$ and $|G(e^{i\theta})| = 1$ a.e., then G is an inner function.

The class $M^{\log}$ can replace N^+ in other contexts.

THEOREM 3. Let $0 < p < \infty$. If $G \in M^{\log}$ and $G(e^{i\theta}) \in L^p$, then $G \in H^p$.

Recently, Kenneth Stephenson has shown that if F and G satisfy the conditions of Theorem 1, then there is an inner function φ and an analytic function ψ from D into D such that

$$F \circ \varphi = G \circ \psi.$$

Some of the ideas underlying Theorem 1 carry over to other domains and higher dimensions. For example, consider a subharmonic function u defined on the half-space $\mathbb{R}^{n+1}_+$, the set of all $z = (x,y)$ with $x \in \mathbb{R}^n$ and $y > 0$. Here let m denote Lebesgue measure on $\mathbb{R}^n$ and

$$N_\alpha(u^+)(x) = \sup\,\{u^+(s,y) : |x - s| < \alpha y\}.$$

THEOREM 4. If u is subharmonic in $\mathbb{R}^{n+1}_+$ with a non-positive nontangential limit superior at almost every $x \in \mathbb{R}^n$ and

$$\liminf_{\lambda\to\infty} \lambda\, m(N_\alpha(u^+) > \lambda) = 0,$$

then $u(z) \leq 0$ for all $z \in \mathbb{R}^{n+1}_+$.

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