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RECENT DEVELOPMENTS IN APPROXIMATE DIFFERENTIATION

1. In recent years a number of authors have studied the fine structure of approximate derivatives. The results of these studies indicate that approximate derivatives possess all the known important properties of derivatives. Three questions arise naturally: 1) What caused this recent surge of activity in a topic that had been a relatively quiet one for some 20 years?; 2) What causes approximate derivatives to behave so much like ordinary derivatives?; and 3) What properties of derivatives are not properties of (all) approximate derivatives?

The first question is easy to answer and we shall answer it. The best we can do with respect to the second question is to offer some insights. All we can do with the third question is pose it as a problem. We know of no such property!

2. In 1950, Zahorski [21] studied the fine structure of (ordinary) derivatives. He obtained a number of far-reaching results, posed some open problems and paved the way for a number of authors to obtain increasingly more delicate results concerning the behavior of derivatives as well as applications of their results. At that time, a great deal was known about the importance of approximate differentiation in various areas of analysis, see, for example, [15], but only a handful of results concerning the fine structure of approximate derivatives were known: if F is approximately differentiable on an interval I, then its approximate derivative is in the first class of Baire and possesses the Darboux property and related properties; furthermore, F is differentiable on a set D containing a dense open set U, and if F'_{ap} is bounded above or below on some interval then F is actually differentiable on that interval. In 1960, Goffman and Neugebauer [7] used interval functions to obtain these known results in an elegant and unified manner. This opened the door for a number of authors to

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Dedicated to Casper Goffman on the occasion of the Conference in Analysis in his honor.

see which of the other properties of derivatives are also properties of approximate derivatives. Because of the work begun by Zahorski concerning the ordinary derivative, and the work [7] by Goffman and Neugebauer, there were many questions to ask about approximate derivatives--and each answer raised new questions.

3. Zahorski [21] studied a hierarchy of classes of functions $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5$. Each of these classes $\mathcal{M}_k$ is defined in terms of the associated sets $E^\alpha = \{x : f(x) < \alpha\}$ and $E_\alpha = \{x : f(x) > \alpha\}$. Membership in a class $\mathcal{M}_k$ involves a requirement of "heaviness" of each associated set near each of its members. As k increases, so does the requirement of heaviness. Thus $f \in \mathcal{M}_0$ if and only if each $x_0 \in E_\alpha$ (or E^α) is a bilateral point of accumulation of E_α (or E^α). For $\mathcal{M}_1$, the operating term is "bilateral point of condensation"; for membership in $\mathcal{M}_2$ we require each unilateral neighborhood of x_0 to intersect E_α or E^α in a set of positive measure. The criteria for membership in $\mathcal{M}_3$ and $\mathcal{M}_4$ are too complicated to state here, (see [21] or [3]); membership in $\mathcal{M}_5$ requires that each point of E_α or E^α be a point of density of that set. Now, Zahorski showed that $\mathcal{M}_0 = \mathcal{M}_1$ = the class of Baire 1 functions possessing the Darboux Property while $\mathcal{M}_5$ consists of the approximately continuous function. Where does the class of derivatives fit into the scheme? It has been known for a long time that this class is contained in $\mathcal{M}_1$. Because of the work of Denjoy [4] it was also known (essentially) that the class of derivatives is contained in $\mathcal{M}_2$. Zahorski showed it is contained in $\mathcal{M}_3$. He also showed that the class of bounded derivatives is contained in $\mathcal{M}_4$. Thus derivatives possess some rather strong properties in terms of associated sets.

It thus became natural to ask whether the more general approximate derivatives possessed similar properties. Membership in $\mathcal{M}_1$ was already known (though not in our present language) because of the work of Tolstoff [17] and Khintchine [8] which was unified by Goffman and Neugebauer [7]. In 1962 S. Marcus [9] showed essentially that each approximate derivative is in $\mathcal{M}_2$, then in 1965 Weil [18] proved membership in $\mathcal{M}_3$. Since a bounded approximate derivative f is in fact an ordinary derivative, such an f must also be in $\mathcal{M}_4$.

In 1973/4, Weil [19] showed that derivatives possessed a property (Z) stronger than that implied by membership in $\mathcal{M}_3$. So do approximate derivatives!

Much more recently, Preiss [14] characterized the associated sets for finite derivatives; for derivatives, possibly infinite, of continuous functions; and for derivatives, possibly infinite, of not necessarily continuous functions. He also proved that the corresponding classes of approximate derivatives have exactly the same family of associated sets. Thus, one cannot distinguish one of these classes of derivatives from the corresponding class of approximate derivatives by associated sets.

Zahorski also obtained a rather delicate monotonicity theorem involving ordinary derivatives and asked whether the analogue using approximate derivatives was valid. It is! ([2], [16]). These theorems did not require differentiability or approximate differentiability <u>everywhere</u>, however. Recently, O'Malley and Weil [13] have proved a theorem which implies that if a condition on the derivative F' of a (everywhere) differentiable function F is sufficient to imply monotonicity of F, then the analogous condition on the approximate derivative G'_{ap} of an approximately differentiable function G will imply monotonicity of G. It would be of interest to know whether the analogous result is valid when one weakens the requirement of differentiability <u>everywhere</u> appropriately.

Finally, we mention recent work concerning the differentiability structure of approximately differentiable function. Such a function is differentiable on a set D containing a dense-open set U. Recent work by Weil, O'Malley, and Fleissner [20], [19], [6], [13], indicate that whatever "wild" behavior F'_{ap} exhibits on I is already exhibited on D. In particular [13], if F'_{ap} takes on the values M and $-M$ on some interval J, then there is a component interval of U on which F' takes on these values. Thus $F'(U \cap J) = F'_{ap}(J)$ for each interval J in the domain of F. (Thus F'_{ap} possesses the Darboux property on

U.) Similarly, Fleissner and O'Malley [6] have shown that F'_{ap} is summable on D. (One cannot replace D with U in this statement.)

4. The results in section 3, above, give an indication of the extent to which approximate derivatives are like derivatives. We turn to our second question--what causes these similarities?

In 1977, O'Malley [12] showed that if F is approximately differentiable on I, then there exists a sequence $\{E_k\}$ of closed sets with union I such that $F'_{ap}|E_k$ can be extended to a derivative on I for each $k = 1,2,\ldots$. This decomposition of F'_{ap} into "pieces of derivatives" allowed several of us [1] to prove that F'_{ap} admits a representation of the form

$$F'_{ap} = g' + hk',$$

where g, h and k are differentiable on I. Now, if the product hk' is itself a derivative F'_{ap} would also be. While hk' need not be a derivative in general (see [5] for a discussion of this subject), it does exhibit a number of properties of derivatives. For example, hk' must be a derivative on each component interval of some dense open set. Thus, the representation $F'_{ap} = g' + hk'$ implies some of the structure of derivatives. This subject is developed in [1]. We note that the term g' cannot be dropped from the representation. In fact there are approximate derivatives F'_{ap} which cannot even be expressed in the form $F'_{ap} = k'_1k'_2\ldots k'_n$ (k'_i a derivative, $i = 1,\ldots,n$).

Another "explanation" for the derivative-like behavior of approximate derivatives was advanced by O'Malley [11]. He developed a very general notion of derivative termed "selective derivative." He showed that each selective derivative possesses a number of properties of ordinary derivatives and he also showed that each approximate derivative can be realized as a selective derivative.

Each of these developments explain in part the desirable behavior of approximate derivatives. But only in part. One would really like a result which could lead one to say "Of course--now I understand the similarities. And here are some more similarities."

One would also like a single nontrivial counterexample to the statement "Every property of derivatives is also a property of approximate derivatives."

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