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The Plane is the Union of Three Rectilinearly Accessible Sets

Call a plane set <u>accessible</u> if through each of its points passes a straight line that does not meet the set again (line of accessibility). Such a set might be thought small, and Banach [1] asked whether a closed accessible set must be of measure zero, but Nikodym [3] constructed an accessible Borel set of full measure in a square, and one of full measure in the plane was constructed in [2].

Another sense in which sets of some given class might be regarded as small would be that the plane cannot be covered by fewer than continuum many sets of the class. It will be shown here, however, that the axiom of choice implies that the plane can be covered by three accessible sets (and these can be made measurable if we assume the continuum hypothesis), although two are insufficient. In fact we shall do a little more. Call a plane set c-<u>densely accessible</u> if through each of its points pass continuum lines of accessibility in every angle.

THEOREM 1. <u>The axiom of choice implies that the plane is the union of three</u> c-<u>densely accessible sets</u>.

<u>Proof</u>. Let $<$ be a well ordering of least possible type of the pairs (p, θ), where p is a point of the plane and θ is a non-degenerate angle with vertex at p. By transfinite induction relative to $<$ we can define for each pair (p, θ)

a line $\ell(p,\theta)$ through p in the angle θ, different from all previously-defined lines, in such a way that (i) if $(q,\Phi) < (p,\theta)$ and $q \neq p$ then $\ell(p,\theta)$ does not pass through q, and (ii) $\ell(p,\theta)$ does not pass through any point of intersection of two previously-defined lines, other than (possibly) p. The transfinite induction can proceed because at any stage there are fewer than continuum many lines to be avoided.

We can assign each point $p \in R^2$ to one of three sets A, B, C by transfinite induction as follows: let (p, θ) be the first pair, relative to $<$, having p as first element. By the construction, there are at most two pairs $(q,\Phi) < (p,\theta)$ such that $\ell(q,\Phi)$ passes through p; we assign p to one of A, B, C to which the first element q of no such pair has been assigned. Evidently the sets A, B, C are c-densely accessible, with the $\ell(p,\theta)$ as the lines of accessibility. The proof is complete.

THEOREM 2. <u>The continuum hypothesis implies that the plane is the union of three measurable</u> c-<u>densely accessible sets</u>.

Since a c-densely accessible Borel set of full measure in the plane was constructed in [2], Theorem 2 is obtained immediately if the following lemma is applied to the complement of this set.

LEMMA. *The continuum hypothesis implies that every plane set* E *of measure zero is the union of two* c-*densely accessible sets*.

Proof. Let $<$ be a well ordering of type ω_1 of the pairs (p,θ), where p is an element of E and θ is a non-degenerate angle with vertex at p. By transfinite induction relative to $<$ we can associate with each pair (p,θ) a line $\ell(p,\theta)$ through p in the angle θ, in such a way that (i) $\ell(p,\theta)\cap E$ has linear measure zero, and (ii) if $(q,\Phi) < (p,\theta)$ then $\ell(p,\theta)$ does not pass through any point of $\ell(q,\Phi)\cap E$ other than (possibly) p. The transfinite induction can proceed because at any stage the directions of the lines to be avoided form a set of measure zero.

We assign each point $p \in E$ to one of two sets A, B by transfinite induction as follows: let (p,θ) be the first pair, relative to $<$, having p as first element. By the construction, there is at most one pair $(q,\Phi) < (p,\theta)$ such that $\ell(q,\Phi)$ passes through p. If there is such a pair, and q has been assigned to A, then we assign p to B; otherwise we assign p to A. Evidently the sets A, B are c-densely accessible, with the $\ell(p,\theta)$ as the lines of accessibility. The proof is complete.

It would be interesting to know whether the axiom of choice could be avoided in Theorem 1, or the continuum hypothesis in Theorem 2 and the Lemma, or indeed whether the plane is the union of three accessible Borel sets. One might also consider the stronger kind of accessibility in which the directions of the lines of accessibility at each point contain a non-empty perfect set in every angle.

The exclusion of a two-set decomposition is entirely elementary.

THEOREM 3. <u>The plane is not the union of two accessible sets</u>.

<u>Proof</u>. Suppose if possible that $R^2 = A \cup B$, and with each point $p \in R^2$ is associated a non-empty set $L(p)$ of lines of accessibility relative to the set (A or B) to which p belongs. Each of A and B must be dense, because otherwise one would have an interior point, but every line through this would meet the set again. Let $a_1, a_2 \in A$, $a_1 \neq a_2$, and let $\ell_1 \in L(a_1)$, $\ell_2 \in L(a_2)$. Let ℓ_1', ℓ_2' be respectively the line through a_1 parallel to ℓ_2, and the line through a_2 parallel to ℓ_1. Let b be an element of B not on any of $\ell_1, \ell_2, \ell_1', \ell_2'$, and let $m \in L(b)$. Now m will meet at least one of $\ell_1 - \{a_1\}$, $\ell_2 - \{a_2\}$, and thus contain a point of B besides b, which is a contradiction, unless m, ℓ_1, ℓ_2 are parallel. In this case choose any point $a_2^* \in m - \{b\}$, so that $a_2^* \in A$, and let $\ell_2^* \in L(a_2^*)$. Now ℓ_2^* is not m, since m contains other points of A, and hence ℓ_2^* is not parallel

to ℓ_1. Replacing a_2, ℓ_2, ℓ_2 by a_2^*, ℓ_2^*, we can repeat the preceding argument and again arrive at a contradiction.

REFERENCES

[1] S. Banach, Problem 32, Fund. Math. 6 (1924), 279.

[2] Roy O. Davies, On accessibility of plane sets and differentiation of functions of two real variables, Proc. Camb. Phil. Soc. 48 (1952), 215-232.

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