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M. Laczkovich and Gy. Petruska, Department I. of Analysis, L. Eötvös University of Budapest, Budapest, Hungary

Differentiable Functions Have Sparse Graphs

Definition 1. *A subset H of the plane is said to be a monotone graph if there exists a monotone function the graph of which can be transformed onto H by a rigid motion (of the plane).*

Definition 2. (see[1]) *A function f(x) is said to have a sparse graph if its graph can be covered by a countable number of monotone graphs.*

The problem of whether or not a differentiable function has sparse a graph was raised in [1], Question 2. It is also stated in [1] that, the graph of an absolutely continuous function is not necessarily sparse. Our theorem 1 answers Question 2 affirmatively (which may be unexpected) and theorem 2 establishes that there is a Lipschitz 1 function with a sparse graph such that its constant multiples do not have sparse graphs. These multiples provide examples of Lipschitz 1 (and hence absolutely continuous) functions with non-sparse graphs; on the other hand they answer Question 10 of [1] in the negative.

Theorem 1. Let the real function $f(x)$ *be defined on a subset* H *of the real numbers and suppose*

(i) H *is everywhere dense in itself;*

(ii) $f'(x)$ *exists at every point of* H ($f'(x) = \pm\infty$ *is allowable*).

Then, $f(x)$ *has a sparse graph.*

Lemma 1. If $|f(x) - f(y)| < |x - y|$ *holds for every* $x,y \in H$, *then the graph of* f *is a monotone graph.*

Proof. Rotate the graph with angle $\frac{\pi}{4}$.

Proof of theorem 1. Let

$$A_{i,n} = \{x \in H: \frac{f(x)-f(y)}{x-y} > 0 \text{ for every } |x-y| < 1/n, y \in H\} \cap [\frac{i-1}{n}, \frac{i}{n}],$$

$$B_{i,n} = \{x \in H: \frac{f(x)-f(y)}{x-y} < 0 \text{ for every } |x-y| < 1/n, y \in H\} \cap [\frac{i-1}{n}, \frac{i}{n}],$$

$$C_{i,n} = \{x \in H: |\frac{f(x)-f(y)}{x-y}| < 1 \text{ for every } |x-y| < 1/n, y \in H\} \cap [\frac{i-1}{n}, \frac{i}{n}]$$

$(i=0,\pm 1,\pm 2,\ldots,n=1,2,\ldots)$.

It is obvious from the differentiability of f that

(i) $H = \bigcup_{i=-\infty}^{+\infty} \bigcup_{n=1}^{\infty} (A_{i,n} \cup B_{i,n} \cup C_{i,n})$,

furthermore for any i,n

(ii) f is increasing on $A_{i,n}$,

(iii) f is decreasing on $B_{i,n}$,

(iv) the graph of $f|_{C_{i,n}}$ is a monotone graph by Lemma 1.

The countable decomposition in (i) and relations (ii) - (iv) prove our theorem.

Theorem 2. There exists a function $f(x)$ *on* $[0,1]$ *such that*

(i) f *satisfies Lipschitz's condition* $|f(x) - f(y)| < |x - y|$ (*in particular its graph is a monotone graph by Lemma* 1).

(ii) *For any* $c > 1$ $cf(x)$ *does not have a sparse graph on* I, *where* I *is any nonempty open subinterval of* $[0,1]$.

We need the following lemma, whose simple proof we omit.

Lemma 2. Let f *be defined on a set* H *which is everywhere dense in itself and suppose that* $f'(x_0) = 0$, $f'(x_1) > 1$, $f'(x_2) < -1$ *hold for some* $x_0, x_1, x_2 \in H$. *Then the graph of* f *is not a monotone graph*.

Proof of theorem 2. We take a decomposition

$$[0,1] = \bigcup_{n=0}^{\infty} H_n$$

with pairwise disjoint measurable and metrically dense subsets H_n (i.e. denoting Lebesgue's measure by $|\cdot|$, we have $|H_n \cap I| > 0$ for every open subinterval $\emptyset \neq I \subset [0,1]$). Put

$$\varphi(x) = \begin{cases} 0, & x \in H_0, \\ 1 - \frac{1}{n}, & x \in H_{2n} \quad (n=1,2,\dots), \\ -1 + \frac{1}{n}, & x \in H_{2n-1} (n=1,2,\dots) \end{cases}$$

and

$$f(x) = \int_0^x \varphi(t)dt.$$

Obviously $|f(x) - f(y)| < |x - y|$ holds for every $x,y \in [0,1]$ and hence by Lemma 1, f has a monotone graph. By Lebesgue's theorem $f'(x) = \varphi(x)$ holds a.e. and hence $f'(x)$ takes the values $0,\ 1 - \frac{1}{n},\ -1 + \frac{1}{n}$ $(n=1,2,\dots)$ almost everywhere on the corresponding subsets H_i.

Suppose that cf has a sparse graph for a given $c > 1$, that is

$$G = \text{graph}(cf) \subset \bigcup_{n=1}^{\infty} \Gamma_n$$

where Γ_n is a monotone graph for every n. We may clearly assume that the monotone function f_n whose graph is Γ_n is defined on the whole real line. Referring to Baire's category theorem there exist N and a subarc $J \subset G$ such that $J \subset \mathrm{cl}\,\Gamma_N$. Let H denote the projection of $J \cap \Gamma_N$ to the axis x.

Obviously, $cf|_H$ has a monotone graph. Since $\mathrm{cl}\Gamma_N \setminus \Gamma_N$ is a countable set, H fills up the interval I corresponding to J apart from countable many points. Therefore $|H_n \cap H| > 0$ holds for every n=0,1,... and hence there exist $x_0 \in H_0 \cap H$, $x_1 \in H_{2n} \cap H$, $x_2 \in H_{2n-1} \cap H$ such that $cf'(x_0) = 0$, $cf'(x_1) = c(1-\frac{1}{n}) > 1$, $cf'(x_2) = c(-1 + \frac{1}{n}) < -1$.

By Lemma 2, the restricted function $cf|_H$ cannot have a monotone graph, a contradiction. The proof is complete.

Remark: With theorem 2 we can answer Question 10 of [1]. Let f denote the function of theorem 2, then putting f with the outer homeomorphism cx $(c > 1)$ the composition cf does not have a sparse graph. Let g(x) denote the inner homeomorphism

$$g(x) = \begin{cases} 2x, & 0 \le x \le 1/3 \\ \frac{1}{2}x + \frac{1}{2}, & 1/3 < x \le 1. \end{cases}$$

Then the composition f(g(x)) does not have a sparse graph, either. Indeed, the graph of f(g(x)) on $[0,\frac{1}{3}]$ is similar to the graph of 2f(x) on $[0,\frac{2}{3}]$ by the similarity tranformation $F(x,y) = (2x,2y)$.

Similarity transformations plainly preserve the sparse graph property. We conclude that the sparse graph property is not an invariant with respect to homeomorphic transformations.

Reference

[1] J. Foran, Continuous Function - A Survey, Real Analysis Exchange, Vol. 2, No. 2 (1977) 85-103.

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