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Summability of Approximate Derivatives

Let $F: [0,1] \to R$ be approximately differentiable with finite approximate derivative F'_{ap}. If $DF = \{x | F'(x) \text{ exists}\}$ and ΔF denotes the interior of DF, then ΔF is a dense open set [2]. Moreover, if F is not everywhere differentiable in the ordinary sense and M is any positive integer, there is a component of ΔF on which F' takes on both M and -M [3]. Thus if F'_{ap} is "well-behaved" on ΔF, one might expect it to be "well-behaved" on [0,1]. For example, if F'_{ap} is bounded above or below on ΔF, then DF=[0,1]. In [1] it is shown that the summability of F'_{ap} over ΔF does not imply its summability over [0,1]. In the positive direction, it is shown that the natural "test set" for summability is $\Delta^* F$, which is the union of all open intervals (a,b) such that F is continuous at each point of (a,b) and F'(x) exists almost everywhere on (a,b). The results and examples are as follows.

Example 1. There exists an unbounded, approximately differentiable function F, such that $F'_{ap}(x)$ is summable over ΔF.

Example 2. There exists on approximately differentiable ACG_* function F, such that $F'_{ap}(x)$ is summable over ΔF but not over [0,1]. Moreover, there is an everywhere differentiable function G such that $F(x) = G(x)$ on $[0,1] \setminus \Delta F$.

Theorem 1. Let $F: [0,1]\to R$ be Baire*1 and Darboux. Let $U(F)=\text{int}\{x: F \text{ is continuous at } x\}$. Suppose F satisfies Lusin's condition (N) on U(F). Let P denote the set $\{x: F' \text{ exists at } x\} \cap U(F)$. Then F is absolutely continuous if and only if the function F' is summable over P.

Corollary 1. Let $F: [0,1]\to R$ have a finite approximate derivative F'_{ap} at each point of [0,1]. Then F'_{ap} is summable over [0,1] if and only if F'_{ap} is summable over $DF = \{x: F' \text{ exists at } x\}$.

Theorem 2. Let $F: [0,1]\to R$ have a finite approximate derivative F'_{ap} at each point of [0,1]. Let Δ^*F be the union of all open intervals I such that

1) F is continuous on I.

2) F is differentiable at almost all x in I.

Then F'_{ap} is summable over [0,1] if and only iff F'_{ap} is summable over Δ^*F.

Corollary 2. Let $F: [0,1]\to R$ have a finite approximate derivative F'_{ap} at all points of [0,1]. Suppose F'_{ap} is summable over ΔF. Then either F is absolutely continuous or there is an open interval (a,b) with

i) $|(a,b)\setminus \Delta F| > 0$.

ii) F is continuous in (a,b).

iii) F is differentiable almost everywhere in (a,b).

References

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