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Symmetric and Ordinary Differentiation

Let f be a real valued function defined on the real line R. The upper symmetric derivate of f at $x \in R$ is

$$D^s f(x) = \limsup_{h \to 0} \frac{f(x+h)-f(x-h)}{2h},$$

and the lower symmetric derivate $D_s f(x)$ of f at x is defined analogously. If $D^s f(x) = D_s f(x)$, then the common value is called the symmetric derivative of f at x, denoted by $f^s(x)$, and in this case we say that f is symmetrically differentiable at x.

In 1927 A. Khintchine [5] proved: a measurable function f: $R \to R$ has a finite ordinary derivative at almost every point x where $D_s f(x) > -\infty$. That the measure zero exceptional set cannot be replaced by a first category set (even for a continuous function) is demonstrated by the function constructed by Z. Zahorski in Lemma III of [8]. (See also Lemma 2 in [7]).

However, our interest here is the improvement of the following interesting consequence of Khintchine's result.

THEOREM K. <u>If the measurable function</u> f: R→R <u>is symmetrically differentiable at each point of</u> R, <u>then</u> f <u>is differentiable in the ordinary sense at almost every point of</u> R.

It was first observed by S. N. Mukhopadhyay [6,Theorem 3] that the exceptional set in Theorem K is also of first category for continuous functions; a simple proof of this using the theory of cluster sets was given by A. M. Bruckner and Casper Goffman [2,p.511]. In [1] the present authors prove the following result:

Theorem 1. <u>If the measurable function</u> f: R→R <u>is symmetrically differentiable at each point of</u> R, <u>then</u> f <u>is differentiable in the ordinary sense at all but a</u> σ-<u>porous set of points in</u> R.

The notion of a σ-porous set was introduced by E. P. Dolženko [3], where he observed that a σ-porous set must be both of measure zero and of first category. L. Zajíček [9] has recently given an example of a perfect set of measure zero which is not σ-porous, thus showing that the σ-porous sets form a proper subclass of the class of all subsets of R that are both of the first category and of measure zero.

J. Foran [4] constructed a continuous function which is symmetrically differentiable at each point of R but which fails to be differentiable in the ordinary sense at uncountably many points. He also posed the question as to whether each perfect set of measure zero could be such an exceptional set. In light of Theorem 1 and the example of Zajíček mentioned in the previous paragraph we can clearly answer this question in the negative.

We close this note by noting that Theorem 1 is obtained by first proving the following more general result, in the statement of which we use the usual notations for the four Dini derivates of f at x.

Theorem 2. *Let* $f: R \to R$ *be a measurable function whose points of continuity are dense in* R. *Then for all but a* σ-*porous set of points* $x \in R$ *both of the following equalities hold*:

$$\text{(i)}\quad D^s f(x) = \max\{D^+ f(x), D^- f(x)\}$$

$$\text{(ii)}\quad D_s f(x) = \min\{D_+ f(x), D_- f(x)\}.$$

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