Real Analysis Exchange Vol. 3 (1977-78)

Clifford E. Weil, Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

The Space of Bounded Derivatives

Let R denote the real line and let

$$D = \{f: R \to R: \ f \text{ is bounded and there is a function } F \text{ such that } f(x) = F'(x) \text{ for all } x \in R\}.$$

For each $f, g \in D$ set

$$d(f,g) = \sup_{x \in R} |f(x) - g(x)|.$$

Then d is a metric on D and convergence in d is uniform convergence. So a standard theorem of advanced calculus says that D is complete. Furthermore, D is a vector space if addition of functions and multiplication of functions by real numbers are defined in the usual way.

As a follow-up to papers on nowhere monotone, differentiabl functions by Goffman and by Katznelson and Stromberg, existence of such functions was established by applying the Baire Category Theorem to an appropriate subspace of D (see [3]). The results here also expand on a paper by Goffman [2] in which he gives a short construction of a bounded derivative which is not Riemann integrable. Another construction can be found on page 26 of the excellent expository article by Bruckner and Leonard [1]. From Theorem 1 it follows, using the Baire Category Theorem, that there are bounded derivatives that are not Riemann integrable on any subinterval of R. Theorem 2 proves even more; namely, that there are bounded derivatives which are discontinuous almost everywhere.

Theorem 1. Let

$$E = \{f \in D: \text{ there is an interval } I \text{ such that } f \text{ is Riemann integrable on } I\}.$$

Then E is of the first category in D.

Proof: Let $\{I_n\}_{n=1}^{\infty}$ be an ordering of the collection of all closed intervals having rational endpoints. For each $n = 1,2,3,\ldots,$ let

$$E_n = \{f \in D: f \text{ is Riemann integrable on } I_n\}.$$

Clearly $E = \bigcup_{n=1}^{\infty} E_n$, and that each E_n is a vector subspace of D. So to prove that E_n is nowhere dense it suffices to show that E_n is closed and $E_n \neq D$.

Suppose $\{f_k\}$ is a sequence of elements of E_n converging in d; that is, uniformly, to f. Since the uniform limit of a sequence of Riemann integrable functions is Riemann integrable, f is Riemann integrable on I_n, and hence $f \in E_n$. By using either the construction of Bruckner and Leonard or of Goffman mentioned above it follows that $E_n \neq D$. This completes the proof.

In [1] (page 27) it is shown that E is the set of discontinuities of a derivative if and only if E is an F_σ, first category set. It follows that there are derivatives which are discontinuous almost everywhere. The important conclusion of Theorem 2 (and also Theorem 1) is that such derivatives are typical. In what follows m denotes Lebesgue measure.

Theorem 2. Let

$$E = \{f \in D: \ m(\{x: f \text{ is continuous at } x\}) > 0\}.$$

Then E is of the first category in D.

Proof: For each $n = 1,2,3,\ldots$ let

$$E_n = \{f \in D: \ m(\{x \in [-n,n]: f \text{ is continuous at } x\}) \geq 1/n\}.$$

Then $E = \bigcup_{n=1}^{\infty} E_n$; so it suffices to show that each E_n is closed and contains no sphere. To accomplish the first objective let $\{f_k\}$ be a sequence in E_n converging in d to f. For each k let

$$H_k = \{x: \ f_k \text{ is continuous at } x\},$$

and let

$$H = \bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} H_k.$$

Since for each k $m(H_k \cap [-n,n]) \geq 1/n$ it follows that $m(H \cap [-n,n]) \geq 1/n$. That f is continuous at each $x \in H$ is a standard $\varepsilon/3$ argument. Consequently, $f \in E_n$.

To show that E_n contains no sphere let $f \in D$ and let $\varepsilon > 0$. There is an $h \in D$ with $|h(x)| \leq \varepsilon$ for all x and $m(\{x: h \text{ is continuous at } x\}) \leq 1/n$. Such a function can be constructed by using the method in [1] or that in [2]. Now for each $0 < \theta < 1$ let $g_\theta = f + \theta h$ and let $B_\theta = \{x: g_\theta$ is continuous at x but h is not$\}$. It is easily seen that B_θ is measurable and $B_{\theta_1} \cap B_{\theta_2} = \emptyset$ if $\theta_1 \neq \theta_2$. It follows that there are at most countably many θ's such that $m(B_\theta) > 0$. Let θ be such that $m(B_\theta) = 0$. Then

$$m(\{x : g_\theta \text{ is continuous at } x\})$$
$$= m(\{x : g_\theta \text{ and } h \text{ are continuous at } x\})$$
$$\leq m(\{x : h \text{ is continuous at } x\}) < 1/n.$$

Thus $g_\theta \notin E_n$, but clearly $d(f, g_\theta) < \varepsilon$. Consequently E_n contains no sphere and the proof is complete.

References

1. A Bruckner, J.L. Leonard, Derivatives, Amer. Math. Monthly, 73 (1966) Part II, 24-56.

2. Casper Goffman, A bounded derivative which is not Riemann integrable, Amer. Math. Monthly, 84 (1977), 205-206.

3. Clifford E. Weil, On nowhere monotone functions, Proc. Amer. Math. Soc., 56 (1976), 388-389.

Received October 6, 1977