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Symmetric Monotonicity

A real valued function f defined on the real line R is said to be nondecreasing at x if there exists a positive number δ_x such that

$$f(x-h) \leq f(x) \leq f(x+h) \text{ for all } 0<h<\delta_x.$$

The function f is said to be symmetrically nondecreasing at x if there exists a positive number δ_x such that

$$f(x-h) \leq f(x+h) \text{ for all } 0<h<\delta_x.$$

We set

$$M = \{x\colon f \text{ is nondecreasing at } x\}$$

and

$$S = \{x\colon f \text{ is symmetrically nondecreasing at } x\}.$$

Clearly $M \subseteq S$. The purpose of this paper is to indicate that S-M is small. "Small", of course, can be interpreted either topologically or measure theoretically and we verify that in either case S-M is "small" for a rather extensive class of functions. In particular, we prove that $|S-M|=0$ ($|\;| \equiv$ Lebesgue measure) if f is a measurable function and that S-M is of the first Baire category if f possesses the Denjoy property; that is, f has the property that for every pair of open intervals I and J

$$I\cap f^{-1}(J) \neq \emptyset \text{ implies } |I\cap f^{-1}(J)|>0.$$

Inherent in our definition is the fact that Denjoy functions are measurable. The class of Denjoy functions contains the class of approximately continuous functions and the class of Baire* 1, Darboux functions recently introduced by R. J. O'Malley [5].

Theorem 1. *If* $f: R \to R$ *is measurable, then* $|S-M|=0$.

Theorem 2. *If* $f: R \to R$ *has the Denjoy property, then* $S-M$ *is of the first Baire category.*

We note that if one employs Theorem 2 it is easy to see that Theorem 1 in [4] remains true for Denjoy functions and, consequently, all fifteen theorems and corollaries stated for continuous functions in [4] remain true for Denjoy functions.

In the last section of our paper we exhibit functions for which S-M is not small in one sense or another. The first two of these examples are characteristic functions of particular additive subgroups of R. Additive subgroups were natural to consider owing to the fact that if f is the characteristic function of such a group G, then $G \subseteq S$. Furthermore, if both G and its complement are dense in R, then M is empty.

Example 1. *There is a function* f *such that* $S-M$ *is both nonmeasurable and of the second Baire category in every interval.*

Notice that this example shows that S may be non-measurable whereas M is known to be measurable for arbitrary functions [2]. It is also, perhaps, interest to note that the function constructed in Example 1 shows that the following theorem of A. Khintchine [3] cannot be extended to arbitrary functions.

Theorem K. _If_ f: R→R _is a measurable function, then_ f _has a finite derivative_ $f'(x)$ _at almost every point at which_ $\overline{f}^s(x)<+\infty$.

$$\left(\overline{f}^s(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}\right)$$

An apparently unpublished theorem of P. Erdös (Búdapest) which was communicated to us by J. Mycielski (Boulder) via J. Foran (Kansas City) proves the existence of the additive subgroup used in Example 2.

Example 2. _If_ $2^{\aleph_0}=\aleph_1$, _then there exists a measurable function_ f: R→R _such that_ S-M _is of the second Baire category_.

It follows from Theorems 1 and 2 that if f: R→R is a Denjoy function, then S-M is both of measure zero and of the first Baire category. Nevertheless, S-M can be uncountable even if f is continuous as our closing example shows.

Example 3. _There is a continuous function_ f: R→R _for which_ S-M _is uncountable_.

References

[1] C. L. Belna, M. J. Evans, P. D. Humke, Symmetric Monotonicity (to appear).

[2] R. O. Davies, L. Eifler, F. Galvin (all independently), Response to Query 9, Real Analysis Exchange 2 (1976), 77-78.

[3] A. Khintchine, Recherches sur la structure des fonctions mesurables, Fund. Math. 9(1927), 212-279.

[4] N. K. Kundu and S. N. Mukhopadhyay, Symmetric derivative of nowhere monotone functions, I. Ann. Pol. Math. 23(1970), 247-253.

[5] R. J. O'Malley, Baire*1, Darboux Functions, Proc. Amer. Math. Soc. (1976), 187-192.

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