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An Alternate Baire Class One Characterization

The purpose of this note is to give a variation of a well-known characterization of Baire class one functions using the class of perfect nowhere dense sets.

Theorem: Suppose f is a function from the interval $[0,1]$ into R. The following statements are equivalent.

1. f is of Baire class one.

2. If H is a perfect subset of $[0,1]$, then $f|H$ contains a point where $f|H$ is continuous.

3. If H is a perfect nowhere dense subset of $[0,1]$, then $f|H$ contains a point where $f|H$ is continuous.

Proof: The equivalence of 1 and 2 is well-known. The implication from 2 to 3 is trivial.

Suppose statement 3. Suppose f is not of Baire class one. By [1] there must exist subsets T and B of $[0,1]$ such that $\mathrm{GLB}f(T) > \mathrm{LUB}f(B)$ and $\mathrm{Cl}(T) = \mathrm{Cl}(B)$.

There exists a sequence of finite collections of open intervals of $[0,1]$, $E_1, E_2, \cdots$, there exists a sequence of finite subsets of T, $J_1, J_2, \cdots$ and there exists a sequence of finite subsets of B, $S_1, S_2, \cdots$ such that if n is a positive integer, the measure of the union of E_n is less than $1/n$, the union of E_{n+1} is a subset of the union of E_n, each element of E_n contains a number in J_n and a number in S_n and if i is a positive integer less than n and x is a number in J_i+S_i, then there is an element of E_n that contains x.

For each n, let D_n denote the closure of the union of E_n. For each n, D_n is closed and bounded. Therefore let D denote the nonempty common part of the sequence $D_1, D_2, \cdots$. For each n, J_n+S_n is a subset of D. D is a perfect nowhere dense set. If P is a point of $f|D$, then $f|D$ is not continuous at P since every open set containing the abscissa of P contains a number in BD and a number in TD. Therefore 3 implies 1.

[1] C.S. Reed, Pointwise Limits of Sequences of Functions, Fundamenta Mathematicae LXVII(1970), pp 183-193.

[2] D.E. Peek, Baire Functions and their Restrictions to special sets, Proceedings American Mathematical Society Vol 30, No 2, October, (1971), pp 303-307.

[3] ________ Pointwise Limits of Sequences of Continuous Functions, Doctoral Dissertation, (1966), University of Texas.

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