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On the Baire class of selective derivatives

The notion of the selective derivative was introduced by R. J. O'Malley (see [1] or [2]). In our paper [3] we solve a problem of O'Malley showing that every selective derivative is of Baire class 2.

Theorem. Suppose that a selection $p_{[x,y]}$ is given (i.e. $x < p_{[x,y]} < y$ holds for every $0 \leq x < y \leq 1$) and the finite selective derivative

$$sf'(x) = \lim_{h \to 0} \frac{f(p_{[x,x+h]}) - f(x)}{p_{[x,x+h]} - x}$$

exists everywhere on $[0,1]$ (for $h<0$ $[x,x+h]$ denotes the interval $[x+h,x]$). Then $sf'(x)$ is of Baire class 2.

Our proof is based on the following

Lemma. Let the interval functions $\ell(x,y)$ and $r(x,y)$ be defined on the subintervals of $[0,1]$ and let $\phi(x,y)$ be a Baire 1 function defined on the set $\{(x,y) : 0 \leq x<y \leq 1\}$ satisfying
$\min(\ell(x,y), r(x,y)) \leq \phi(x,y) \leq \max(\ell(x,y), r(x,y))$
for every $0 \leq x<y \leq 1$. If the limits

$$\lim_{y \to x-0} \ell(y,x) = \lim_{y \to x+0} r(x,y) = g(x)$$

exist for every $0<x<1$, then the function $g(x)$ is of Baire class 2 on $(0,1)$.

By a theorem of O'Malley, $f(x)$ is Baire 1 ([1], Theorem 10); thus the function $\phi(x,y) = \frac{f(x)-f(y)}{x-y}$ is Baire 1 on the set $\{(x,y): 0\le x<y\le 1\}$. We put

$$(1)\qquad \ell(x,y) = \frac{f(y) - f(p_{[x,y]})}{y - p_{[x,y]}} \quad (0\le x<y\le 1),$$

$$(2)\qquad r(x,y) = \frac{f(p_{[x,y]}) - f(x)}{p_{[x,y]} - x} \quad (0\le x<y\le 1);$$

then by definition $\lim_{y\to x-0} \ell(y,x) = sf'(x)$ $(0<x\le 1)$ and $\lim_{y\to x+0} r(x,y) = sf'(x)$ $(0\le x<1)$.

It is easy to see that for every selection $p_{[x,y]}$ we have

$$(3)\qquad \min(\ell(x,y),r(x,y)) \le \frac{f(y)-f(x)}{y-x} \le \max(\ell(x,y),r(x,y)).$$

Hence the lemma is applicable and the theorem is proved.

The formulas (1), (2) and (3) lead to the following generalisation of selective derivatives.

Definition. Let $f(x)$ be an arbitrary function on $[0,1]$. Suppose that the interval functions $\ell(x,y)$ and $r(x,y)$ are defined on the subintervals of $[0,1]$ and satisfy (3). If the finite limits $\lim_{y\to x-0} \ell(y,x)$ and $\lim_{y\to x+0} r(x,y)$ exist and are equal, then $f(x)$ is said to be differentiable at the point x with respect to $\ell(x,y)$ and $r(x,y)$ and the derivative ${}^{r}_{\ell}f'(x)$ is defined by

$${}^{r}_{\ell}f'(x) = \lim_{y\to x-0} \ell(y,x) = \lim_{y\to x+0} r(x,y).$$

The following theorems are proved.

1) If f(x) is differentiable with respect to both $\ell_1(x,y)$, $r_1(x,y)$ and $\ell_2(x,y)$, $r_2(x,y)$, then ${}^{r_1}_{\ell_1}f'(x)={}^{r_2}_{\ell_2}f'(x)$ holds on [0,1] apart from a countable set.

2) If f(x) is differentiable with respect to $\ell(x,y)$ and $r(x,y)$ and ${}^{r}_{\ell}f'(x)>0$ for every $x\in[0,1]$, then f(x) is non-decreasing on a subinterval of [0,1]. If in addition f(x) is a Darboux function then f(x) is non-decreasing on [0,1].

3) Suppose that f(x) is differentiable on [0,1] with respect to $\ell(x,y)$ and $r(x,y)$. Then f(x) is Baire 1 and there exists an everywhere dense open set U such that f(x) is continuous and almost everywhere differentiable on U.

4) If f(x) is differentiable on [0,1] with respect to $\ell(x,y)$ and $r(x,y)$, then the set of points of continuity of the (ordinary) derivate numbers $\underline{f}$ and $\overline{f}$ is everywhere dense in [0,1].

5) If f(x) has the selective derivative sf'(x) for a given selection, then the set of points of continuity of sf'(x) is everywhere dense in [0,1].

6) If f(x) is differentiable on [0,1] with respect to $\ell(x,y)$ and $r(x,y)$, then there is a set $H\subset[0,1]$ such that f(x) is differentiable at the points of H, ${}^{r}_{\ell}f'(x)=f'(x)$ holds for every $x\in H$, and $[0,1]\setminus H$ is of the first category.

7) If f(x) is differentiable on [0,1] with respect to $\ell(x,y)$ and $r(x,y)$, then the function ${}^{r}_{\ell}f'(x)$ is Baire 2.

Problem: Suppose that ${}^{r}_{\ell}f'(x)$ exists on [0,1]. Does the function ${}^{r}_{\ell}f'(x)$ belong to the family of the honorary functions of the second class (i.e. there is a function g(x) in the first Baire class such that ${}^{r}_{\ell}f'(x)=g(x)$ except on a countable set)? Is it true for the selective derivatives? We note that O'Malley has constructed a selective derivative which is not Baire 1 but his function is an honorary function of the second class.

References

[1] R. J. O'Malley, Selective derivatives, Acta Math. Acad. Sci. Hungar., to appear.

[2] ____________, Selective derivatives, Real Analysis Exchange, vol. 1, 1/1976/,50-51.

[3] M. Laczkovich, On the Baire class of selective derivatives, Acta Math. Acad. Sci. Hungar., to appear.

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