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Totally and Partially Ambiguous Points of Planar Functions

Let f be a function from the plane to the complex sphere. A point $z \in P$ is an (rectilinearly oppositely) ambiguous point of f with $\Theta \in [0,\pi)$ as direction of ambiguity if there are two linear arcs, Λ_1 and Λ_2, at z with directions Θ and $\Theta-\pi$, resp., such that $Cl(\Lambda_1,z) \cap Cl(\Lambda_2,z)$ is empty. ($Cl(\Lambda,z)$ is the cluster set of f at z along the arc Λ.) Let $f \in A$ if every point of the plane is an ambiguous point of f. H. Fox[2, Theorem 20] exhibits a function f in the set A with the range an enumerable nowhere dense set.

For a given function f there may be more than one direction of ambiguity at a point p. The following two theorems give some examples.

Theorem 1. *There exists $f \in A$ such that f has enumerably many distinct directions of ambiguity at every point of the plane. Moreover the range of f is a bounded nowhere dense subset of the real line with measure zero.*

Theorem 2. *There exists a function $f \in A$ such that f has an everywhere dense set of directions of ambiguity at every point of the plane. The range of f again is a* bounded nowhere dense subset of the real line with measure zero.

However the next theorem shows that a countable number of directions at each point is the most possible.

Theorem 3. The set of points $p \in P$ such that uncountably many directions at p are directions of ambiguity of f is a sparse set.

The definition of sparse set is given in [1].

Next, totally ambiguous points of functions are investigated. If $f \in A$ then the set of totally ambiguous points of the function f, denoted by T(f), is the set of points of the plane that have every direction as a direction of ambiguity. By Theorem 3, T(f) is a sparse set. The following three theorems give examples of functions $f \in A$ for which T(f) has various properties.

Theorem 4. Given T a subset of the plane with $|T| \leq \aleph_0$, there exists an $f \in A$ such that $T \subseteq T(f)$.

The set T(f) can also be big in the sense of cardinality.

Theorem 5. There exists an $f \in A$ with $|T(f)| = 2^{\aleph_0}$. Moreover the range of f can be countable.

Theorem 6. If $2^{\aleph_0} = \aleph_1$, there exists an $f \in A$ with $|T(f)| = 2^{\aleph_0}$ and the range of f consists of 8 points.

In addition to being sparse, T(f) has a stronger property, called supersparse, which will be defined next.

Definition. Let ρ_1 and ρ_2 be in the interval $[0,\pi)$ and assume ρ_1 is less than ρ_2. A set T is called (ρ_1,ρ_2)-void if the angle of the line joining any two points of T does not lie in the interval (ρ_1, ρ_2).

Definition. A set S is supersparse if for every rational interval (ρ_1, ρ_2) contained in $[0,\pi)$, S can be decomposed into an at most countable union of sets S_j with the property that there exists a subinterval (ρ_1^j, ρ_2^j) of (ρ_1, ρ_2) for which S_j is (ρ_1^j, ρ_2^j)-void.

Theorem 7. There exists f∈A such that the set of points in the plane at which f has uncountably many directions as directions of ambiguity is not a supersparse set.

However the next theorem shows that if the set of partially ambiguous points is restricted further the result is a supersparse set.

Theorem 8. Let f∈A. If B is the set of points in the plane at which f has all but a nowhere dense set of directions as directions of ambiguity, then B is supersparse.

Corollary. For every f∈A, T(f) is supersparse.

The next question that arises is whether every supersparse set can be a T(f) for some f. This is not always true. The following theorem though does apply for any supersparse set.

Theorem 9. If a set S is supersparse, there exists a function f∈A such that for every z∈S, z is an ambiguous point of f for all but a nowhere dense set of directions.

Some facts to be noted about supersparse sets are that they exist(see Theorems 4 and 5) and they are sparse; however we show that there are sparse sets that are not supersparse.

REFERENCES

1. H. Blumberg, Exceptional sets, Fund. Math. 32 (1939), 3-32

2. H. Fox, The continuum hypothesis and planar functions, Ph.D. Thesis, University of Wisconsin-Milwaukee, 1972.

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