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The Symmetric and Ordinary Derivative

It is known that if $f(x)$ is a continuous real valued function whose symmetric derivative $f_s'(x)$ exists everywhere, then $f'(x)$ exists except on a set which is both of first category and of measure 0. The most commonly occurring functions whose symmetric derivative exists everywhere are those which have finite right and left derivatives (such as functions which are the difference of two concave upward functions) and then the exceptional set where $f'(x)$ does not exist is at most countable. That the exceptional set need not be countable is shown by the example below. However, the exceptional set of this example is 'small', i.e., of Hausdorff dimension 0 (it is an enumerable set along with a perfect set which can be covered with 2^{n-1} intervals of size $2^{-n}(n!)^{-2}$). It is reasonable to ask whether the dimension of the exceptional set can be increased or whether any perfect set of measure 0 can be the exceptional set.

<u>Example</u>: There is a continuous function $f(x)$ such that $f_s'(x)$ exists everywhere but $f'(x)$ fails to exist on an uncountable set.

Construction: The exceptional set E for f is constructed by dividing $[0,1]$ into 8 equal subintervals and selecting two of them. In general, each interval at the nth stage is divided into $2(n+1)^2$ equal subintervals of length $2^{-(n+1)}((n+1)!)^{-2}$ and two subintervals are selected in each interval of the nth stage. The points which are in infinitely many of these intervals form a perfect set.

Specifically, let A_n be the set of numbers in $[0,1]$ of the form $k(n!)^{-2}2^{-n+1}$ where k is an integer. Let $E_1 = [0,1]$ and for $N \geq 2$, let E_N be the collection of all real numbers $x = \sum_2^\infty k_n(n!)^{-2}2^{-n+1}$ where $0 \leq k_n < 2n^2$ and, furthermore, for each $n \leq N$ $k_n = n^2-n$ or $k_n = n^2+n$. Let $E = \cap E_n$. For each natural number $n \geq 2$, define:

$$f_n(x) = \begin{cases} \text{dist}(x,A_n) & \text{if } x \in E_{n-1} \setminus E_n \\ 0 & \text{otherwise.} \end{cases}$$

It is readily observed that each $f_n(x)$ is continuous and that $f_n(x) \leq \frac{1}{2}(n!)^{-2}2^{-n+1}$. Thus $f(x) = \sum_2^\infty f_n(x)$ is continuous. If $x \notin E$, then $f_s'(x)$ exists, since both of the one-sided derivatives of f exist. Let $x \in E$. Then $D_+f(x) = 0$ and $D^+f(x) \geq 1/3$. To see this, given $\varepsilon > 0$, choose N so that $2(N!)^{-2}2^{-N+1} < \varepsilon$ and choose k_n so that $0 \leq k_n < 2n^2$ and $x \in [a,a+h]$ where $a = \sum_2^N k_n(n!)^{-2}2^{-n+1}$ and $h = (N!)^{-2}2^{-N+1}$. Let

$h_1 = a+h-x$, $h_2 = a+h-x+\frac{1}{2}h$. Then $f(x) = f(x+h_1) = 0$ and $f(x+h_2) = \frac{1}{2}h$. Thus $\frac{f(x+h_1) - f(x)}{h_1} = 0$, $\frac{f(x+h_2) - f(x)}{h_2} = \frac{\frac{1}{2}h}{h_2} \geq 1/3$ since $h_2 \leq 3/2\ h$. Thus $f'(x)$ does not exist. It remains to show that $f'_s(x)$ does exist.

Given $h > 0$, suppose $(n!)^{-2}2^{-n+1} \leq h$. $\leq ((n-1)!)^{-2}2^{-n+2}$. If $x+h$ and $x-h$ belong to $E_{n-1} \setminus E_n$, then $|f(x+h)-f(x-h)| = |f_{n-1}(x+h)-f_{n-1}(x-h)|$ which is less than or equal to twice the distance of x to the center of the interval in E_{n-1} which x belongs to. That is, $|f(x+h) - f(x-h)| < 2n(n!)^{-2}2^{-n+1}$ and $h > \frac{1}{2}(n^2-2n)(n!)^{-2}2^{-n+1}$. Thus

$$\left|\frac{f(x+h) - f(x-h)}{2h}\right| < \frac{2n}{n^2-2n} .$$

If exactly one of $x+h$, $x-h$ belongs to E_{n-1} , say $x+h$ does, then

$$\tfrac{1}{2}(n^2-2n-1)(n!)^{-2}2^{-n+1} \leq h \leq \tfrac{1}{2}(n^2+n)(n!)^{-2}2^{-n+1}$$

and

$$f(x+h) = f_{n-1}(x+h) \leq (2n+1)(n!)^{-2}2^{-n+1}$$

because $x+h$ extends at most this far out of the interval of E_n which contains x .

Since $f(x-h) = f_n(x-h) \leq \frac{1}{2}(n!)^{-2}2^{-n+1}$,

then

$$\left|\frac{f(x+h) - f(x-h)}{2h}\right| \leq \frac{2n + 3/2}{n^2-2n-1} .$$

Suppose both $x+h$ and $x-h$ belong to E_n. If $h > n(n!)^{-2}2^{-n+1}$, then, since $f(x+h)$ and $f(x-h)$ are both less than $n!2^{-n+1}$,

$$\left|\frac{f(x+h) - f(x-h)}{2h}\right| < \frac{1}{2n} .$$

On the other hand, if $h \leq n(n!)^{-2}2^{-n+1}$, then $|f(x+h) - f(x-h)|$ is less than or equal to twice the distance of x to the center of the interval in E_n which contains x. That is

$$|f(x+h) - f(x-h)| \leq 2(n+1)(n+1)^{-2}2^{-n}$$

and

$$\left|\frac{f(x+h) - f(x-h)}{2h}\right| \leq \frac{2(n+1)(n+1)^{-2}2^{-n}}{2 \cdot (n!)^{-2}2^{-n+1}} = \frac{2}{(n+1)} .$$

These represent all possible cases.

Now, as $h \to 0$, $n \to \infty$ and $\frac{f(x+h) - f(x-h)}{h} \to 0$; thus $f'_s(x) = 0$.

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