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CONTINUOUS FUNCTIONS

- A SURVEY -

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The purpose of this survey is to consider some of the broad classes of continuous functions and bring together some of the known relationships between these classes. Generally, the properties which define the classes to be considered will be relevant to integration and have been thought of as being properties which an indefinite integral should possess. Throughout the discussion which follows, all functions under consideration will be presumed to belong to C = the class of continuous real valued functions defined on a closed interval.

Functions of bounded variation, BV, and absolutely continuous functions, AC, are two of the most important classes of continuous functions. Functions of bounded variation are differentiable almost everywhere and their derivatives are Lebesgue integrable. Absolutely continuous functions are the primitives for the Lebesgue integral. These classes are closed under both addition and multiplication. If, however, composition of functions is also allowed, it is surprising how readily any continuous function can be represented by means of combinations of absolutely continuous functions and functions of bounded variation. The possibilities for such combinations were determined most completely in the memoir of Nina Bary [2]. She showed that any continuous function F can be represented as

$$h_1 o g_1 + h_2 o g_2 + h_3 o g_3$$

where the h_i and g_i are absolutely continuous. Furthermore, the h_i can be chosen so as to be homeomorphisms (i.e., strictly monotone) and a sum containing fewer than three compositions will not suffice in general. In fact, functions which are differentiable on a set of positive measure in every interval of their domain are characterized as being representable as a sum of two of these compositions. Moreover, any function which can be written as the composition of three absolutely continuous functions is shown to be also representable as the composition of two absolutely continuous functions. If the g_i are only required to be of bounded variation and the h_i are to be absolutely continuous homeomorphisms, then any continuous function is shown to be expressible as the sum of two compositions. It is further shown that for each continuous function F there exist homeomorphisms h_1 and h_2 so that $F o h_1$ and $h_2 o F$ are differentiable almost everywhere. The question posed by Nina Bary in this memoir as to whether every continuous function could be represented as $f o g o h$, where f , g , and h are continuous functions of bounded variation is answered in the negative by her in [3].

Here the situation is more complicated. Let BV be the class of continuous bounded variation functions. For isolated ordinals $\alpha+1$, let $BV_{\alpha+1}$ be all functions of the form $h \circ g$ which do not belong to any BV_β , $\beta < \alpha$, where $h \in BV$ and $g \in BV_\alpha$. For limit ordinals λ let BV_λ be the class of continuous functions f which do not belong to any BV_β , $\beta < \lambda$, such that f is the limit of some sequence of the form

$$\{f_1 \circ g , f_2 \circ f_1 \circ g , \ldots f_n \circ f_{n-1} \circ \ldots \circ f_1 \circ g \ldots\}$$

where g belongs to some BV_β , $\beta < \lambda$, and each of the f_i are of bounded variation. Then it is shown that each of the classes BV_α , $\alpha < \omega_1$, are non-empty and that there are continuous functions which do not belong to any of the classes BV_α . Additional results of this type can be found in [17] and [18].

Definitions and Notation

In order to be concise in what follows, H will denote the class of homeomorphisms and $\bar{H}$ will denote the class of absolutely continuous homeomorphisms. A general interval in the range of a function f will be denoted by J , in the domain by I , a set of measure 0 by Z . Thus, wherever J , I , or Z are used to describe a class of functions, "$\forall J$ in the range of f," "$\forall I$ in the domain of f," and "$\forall$ set Z of measure 0" is to be understood to preceed the description. The set of points x where f' exists and is finite will be denoted by D, the set of points x where f' exists finite or infinite by D', the relative complement of D with respect to the domain of f by N, the complement of D' with respect to the domain of f by N'. The Lebesgue measure of a set E will be denoted by $|E|$ and the cardinality of E by $\|E\|$. Note that $|D| = |D'|$ always holds (cf. e.g., [16, p.267]). We will write $A \circ B$ for the class of functions $F = g \circ h$, $g \in A$, $h \in B$.

Several properties of functions are defined in terms of level sets and hence the resulting classes of functions are invariant under the application of an inner homeomorphism. For example,

T_1: $f \in T_1$ provided $|\{y: f^{-1}(y)$ is infinite$\}| = 0$

T_2: $f \in T_2$ provided $|\{y: f^{-1}(y)$ is uncountable$\}| = 0$

B_1: $f \in B_1$ provided $|f(N) \cap J| < |J|$

B_2: $f \in B_2$ provided $\{y: f^{-1}(y)$ is finite$\} \cap J$ is uncountable.

T_1 and T_2 were introduced by Banach in [1]. The symbols B_1 and B_2 are used here for properties introduced by Nina Bary in [2]. There it is shown that $H \circ BV = B_2$ [2, p.635] and that $f \in B_1$ iff $\exists g \in \bar{H}$,

$\phi \in AC$ such that $\phi = g \circ f$. The class BV along with the classes BVG^* , BVG , MG defined below are also invariant under the application of an inner homeomorphism.

Additional properties such as Lusin's condition (N) and Banach's condition S indicate a relationship between sets in the domain and sets in the range of a function. We will need:

(N): $f \in (N)$ provided $|f(Z)| = 0$

S: $f \in S$ provided $\forall \varepsilon > 0 \; \exists \delta > 0 \ni |E| < \delta \Rightarrow |f(E)| < \varepsilon$

S′: $f \in S'$ provided $\forall J \; \exists \varepsilon > 0 \ni J \subset F(E) \Rightarrow |E| > \varepsilon$.

The classes ACG^* , BVG^* , ACG , BVG and MG are in frequent use because of their relationship to the Denjoy integral. Although it is not done in this paper, nice sufficient conditions for a function to belong to MG in terms of level sets should be obtainable (cf. [9]).

ACG^*: $f \in ACG^*$ provided the domain of f can be written as a countable union of sets E_n such that for each n and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if I_j are any sequence of pairwise non-overlapping closed intervals with endpoints in E_n and $\Sigma|I_j| < \delta$ then $\Sigma \mathcal{O}(f;I_j) < \varepsilon$. Here

$$\mathcal{O}(f;I_j) = (\sup f(x) - \inf f(x'))_{x,x' \in I_j} .$$

BVG*: $f \in$ BVG* provided the domain of f can be written as a countable union of sets E_n such that for each n there exists M_n so that if I_j are any sequence of pair-wise non-overlapping closed intervals with endpoints in E_n, then $\Sigma\mathcal{O}(f;I_j) < M_n$.

ACG: $f \in$ ACG provided f satisfies the definition given for ACG* with $\mathcal{O}(f;I_j)$ replaced by $|f(b_j) - f(a_j)|$ where $I_j = [a_j,b_j]$.

BVG: $f \in$ BVG provided f satisfies the definition given for BVG* with $\mathcal{O}(f;I_j)$ again replaced by $|f(b_j) - f(a_j)|$.

MG: $f \in$ MG provided the domain of f can be written as a countable union of sets E_n and, for each n, f is monotone on E_n.

Finally, the following abbreviations will be used. They represent properties which tend to be intermediate to the above classes and the classes of functions so described tend to be elusive and of doubtful importance on their own.

SpB: $f \in SpB$ provided f has a sparse graph; i.e., the graph of f can be contained in the graphs of a sequence of functions, each defined and monotone with respect to their own (possibly different) coordinate axes.

$\sigma.f.\ell.$ $f \in \sigma.f.\ell.$ provided the graph of f is of σ-finite linear measure.

$\Lambda N=0$: $f \in \Lambda N=0$ provided the linear measure of the graph of f on N is 0 .

$\Lambda N'=0$: $f \in \Lambda N'=0$ provided the linear measure of the graph of f on N' is 0 .

$\Lambda Z=0$: $f \in \Lambda Z=0$ provided the linear measure of the graph of f on sets of measure 0 is always 0 .

Equivalence of Classes of Functions

Several of these classes of functions and those obtained by forming compositions with H or $\overline{H}$ are equivalent. The chart below shows equivalences that are known. Here, 'diff.' refers to the class of differentiable functions.

Equivalence	Reference for the Proof
1) AC $\Leftrightarrow$ BV and (N)	[16, p.227]
2) ACG* $\Leftrightarrow$ BVG* and (N)	[16, p.233]
3) ACG $\Leftrightarrow$ BVG and (N)	[16, p.227]
4) $\|F(N')\|=0 \Leftrightarrow AC\circ AC \Leftrightarrow T_1$ and $(N) \Leftrightarrow \overline{H}\circ AC$	[16, p.284-289]
5) $H\circ BV \Leftrightarrow B_2$	[2, p.635]
6) diff $\circ$ H $\Leftrightarrow$ BVG*	[4]
7) $AC\circ H \Leftrightarrow BV$	[4]
8) H $\circ$ diff $\Leftrightarrow$ S′	[6]
9) $\|D \cap I\| > 0 \Leftrightarrow AC\circ AC + AC\circ AC$	[2, p.222]

Material in addition to that in [16] can be found in [17]. In 4) reference for the proofs is made to [16], whereas most of the results originate in [2]. The result in 7) is a corollary to the theorem that states: for every function f of bounded variation, there is a homeomorphism g such that f∘g satisfies a Lipschitz condition.

In trying to systématize the relationships between the main classes, several questions arose and appear to be unanswered. These will be included in the text of this survey. E.g., in [19], Tolstov proves that every ACG* function can be transformed by a suitably chosen inner differentiable homeomorphism into a differentiable function. This leaves open the question as to whether the class ACG* and diff $\circ$ $\overline{H}$ coincide, that is

Question 1. Can every ACG* function be written as f∘g where f is differentiable and g is monotone and absolutely continuous?

The problem here arises from the fact that the inverse of a function in $\bar{H}$ need not be in $\bar{H}$.

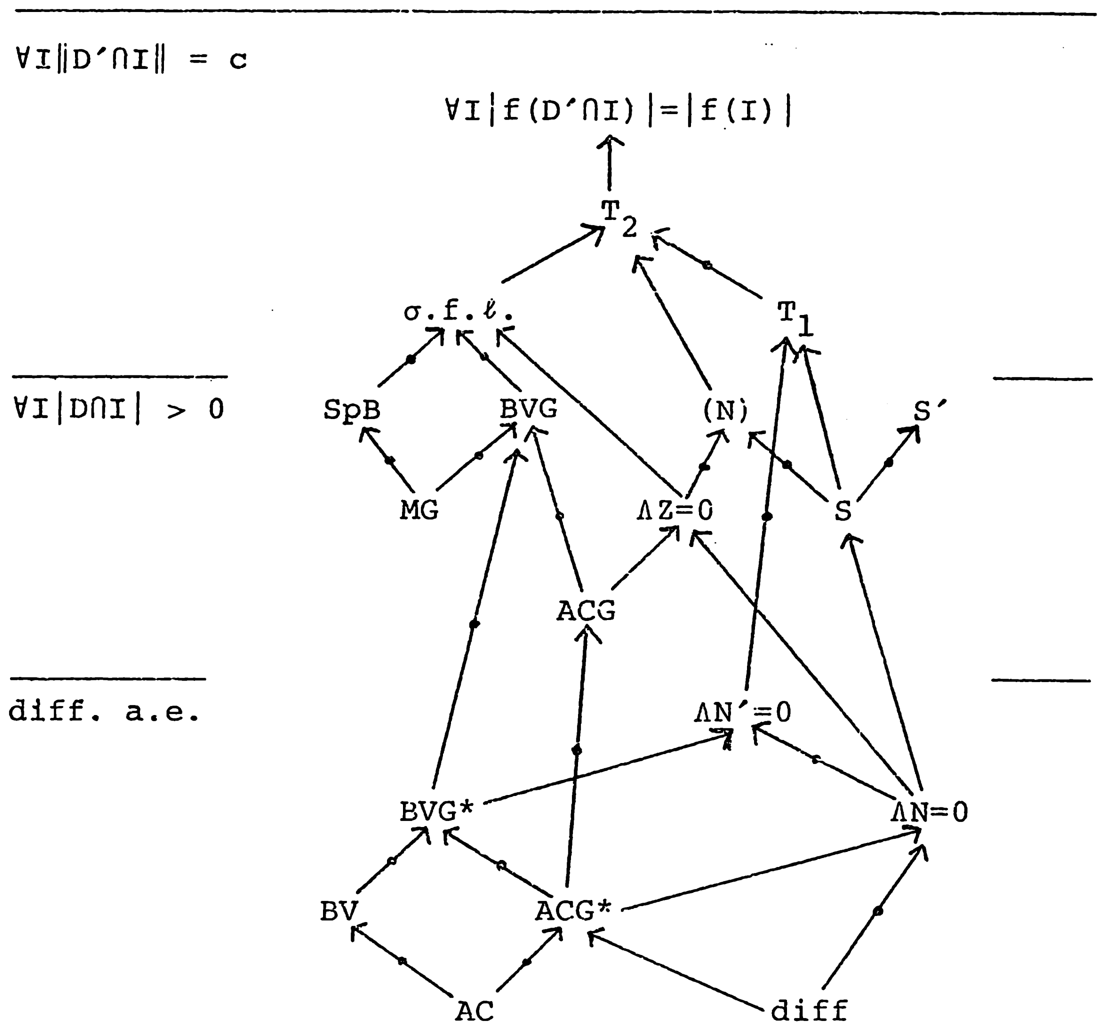
Implications Between Classes

The diagram below shows how these properties are related and the differentiation characteristics of the properties. No implications other than those shown and those that can be obtained by following the arrows hold between the various classes shown on the diagram (equivalent classes are represented only once on the diagram) with the possible exception given by

Question 2. Does there exist a differentiable function whose graph is not sparse?

It is known that there are differentiable functions which are not monotone on any interval. This suggests an affirmative answer to this question.

Diagram of Simple Implications



The arrows which correspond to easily obtained implications are marked with a dot. In those cases, all that is required for the proof of the implication is either the use of the definition or the argument that the projection on either axis of a set of linear measure 0 is of measure 0 . Thus, e.g., $\Lambda N' = 0$ implies $|F(N')| = 0$, which is equivalent to T_1 .

From the fact that a function which is AC satisfies $\Lambda Z = 0$, ACG implies $\Lambda Z = 0$ is easily obtained.

Functions which are ACG* are BVG* , and both classes are differentiable almost everywhere [16, p. 230]. Then, for ACG* functions $|N| = 0$ and hence $\Lambda N = 0$.

Any function f is ACG* on the set of points where f' exists and is finite [16, p.235]. Hence $\Lambda N = 0$ implies $\Lambda Z = 0$. For if $|Z| = 0$, the length of the graph of f on $Z \cap D$ is 0 , and by hypothesis, the length of the graph of f on $Z \cap N$ is also 0 .

The remaining implications are given below:

Implication	Reference for Proof
diff $\Rightarrow$ ACG*	[16, p.235]
BVG* $\Rightarrow \Lambda N' = 0$	[16, p.230]
$\Lambda N = 0 \Rightarrow S$	$\Lambda N=0 \Rightarrow \|F(N)\|=0 \Rightarrow S$
$S \Rightarrow T_1$	$S \Leftrightarrow T_1$ and (N)
$N \Rightarrow T_2$	[1] also [16, p.286]
$\Lambda Z = 0 \Rightarrow \sigma.f.\ell.$	[8]
$\sigma.f.\ell. \Rightarrow T_2$	[7]
$T_2 \Rightarrow \|f(D' \cap I)\| = \|f(I)\|$	[16,280]

That the differentiability 'levels' on the diagram of implications are best possible can be seen as follows:

i) The condition $\Lambda N' = 0$ implies $|N'| = 0$, but $|N'| = |N|$, and thus these functions must be differentiable almost everywhere.

ii) An example of a function which is MG and ACG but not differentiable almost everywhere is given in [16, p.224].

iii) An example of a function in S which is not differentiable almost everywhere is given in [12].

iv) Condition (N) implies $|D \cap I| > 0$ can be found in [1] also [16, p.286].

v) That S′ implies $|D \cap I| > 0$ is shown in [6].

vi) For functions BVG or SpB , the Baire Category Theorem implies that within each interval I there is a subinterval I′ on which the function is BV or monotone with respect to some axis, respectively. Then $|D \cap I| > 0$ follows from differentiability almost everywhere on I′.

vii) An example of a function which satisfies T_1 and is of σ-finite length but does not have a finite derivative at any point is provided in [7].

viii) Clearly, $|f(D' \cap I)| = |f(I)|$ implies $\|D' \cap I\| = c$.

To show that there are no more implications than those indicated on the diagram is obviously a task of example construction. An example of a function in (N) which does not satisfy S is given in [5]; an example of a function in S' which is not in T_2 is given in [6]. Most of the other examples are well known, but a few require some ingenuity. The task is simplified by showing that no arrow on the diagram can be reversed, then using proven levels of differentiability and distinguishing between those functions which must satisfy (N) and those that need not satisfy (N). Providing all the examples needed would take more space than is justified.

Additional classes of continuous functions could, of course, be placed on the diagram. For example, $N' = \phi$, the class of functions which are everywhere differentiable in the extended sense is intermediate to diff and $\Lambda N' = 0$ and implies BVG*. The first two implications are immediate and the third follows from Theorem (10.1), [16, p. 234]. The only missing arrows for which the counterexample is not trivial are that $N' = \phi$ implies neither $\Lambda N = 0$ nor ACG*. An example which shows that neither of these implications holds can be found in [16, pp. 205-206].

Influence of Homeomorphisms on the Classes

It is perhaps academic in some cases to ask what happens to a given class of functions when transformed by inner or outer homeomorphisms. Nonetheless, the following results are known with respect to outer homeomorphisms

$$HoAC = HoACG^* = Hodiff = HoS = HoS' = S' .$$

These follow from the fact that $\overline{H}oAC = S$, diff $\subset$ ACG* $\subset$ S , and Hodiff = S′ , as stated previously.

$$HoBV = HoBVG^* = HoT_1 = B_2$$

These follow from the fact that HoBV = B_2 and $\overline{H}oAC = T_1$, as stated previously. What happens to the remaining main classes appears to be unknown. Thus, the following questions arise:

Question 3. How can the class of functions of the form fog , where f is a homeomorphism and g is ACG , be characterized?

Question 4. How can the class of functions of the form fog , where f is a homeomorphism and g is BVG be characterized?

Question 5. How can the class of functions of the form fog , where f is a homeomorphism and g satisfies Lusin's condition (N) be characterized?

Question 6. How can the class of functions of the form fog , where f is a homeomorphism and g satisfies Banach's condition T_2 be characterized?

If the outer homeomorphisms are to be absolutely continuous, then

$$\bar{H}oAC = \bar{H}oACG^* = \bar{H}oS = S$$

and

$$\bar{H}oBV = \bar{H}oBVG^* = \bar{H}oT_1 = T_1$$

follow from what has been noted previously. Clearly, $\bar{H}o(N) = (N)$ and $\bar{H}oT_2 = T_2$. This leaves open the following questions:

Question 7: How may the functions of the form fog , where f is an absolutely continuous homeomorphism and g is differentiable, be characterized?

Question 8: How may the functions of the form fog , where f is an absolutely continuous homeomorphism and g is ACG , be characterized?

Question 9: How may the functions of the form fog , where f is an absolutely continuous homeomorphism and g is BVG , be characterized?

With respect to these last three questions, one has $\bar{H}odiff \subset (N)$, $\bar{H}oACG \subset (N)$ and $\bar{H}oBVG \subset T_2$. Perhaps, identity holds.

Functions transformed by inner homeomorphisms tend to behave nicely with respect to integration theory, providing the transformation yields a primitive (cf. e.g., [10] and [11]). With respect to inner homeomorphisms

$$AC \circ H = BV \circ H = BV$$

and

$$diff \circ H = ACG^* \circ H = BVG^* \circ H = BVG^*$$

as stated previously. It has already been noted that BVG, MG, T_1, T_2, B_1, and B_2 are unchanged by inner homeomorphisms. Of interest, perhaps, is the following question:

Question 10. If $f(x)$ has a sparse graph and g is a homeomorphism, must both $g \circ f$ and $f \circ g$ have sparse graphs?

No new questions arise from consideration of inner absolutely continuous homeomorphisms with respect to the main categories on the diagram.

Influence of Addition on the Classes

It was shown early [14] that the addition of a linear function to one satisfying Lusin's condition (N) need not satisfy condition (N). The classes AC, BV, ACG*, BVG*, ACG, and BVG are all closed under addition. Nina Bary showed that any continuous function can be written as the sum of two T_1 functions or three functions in S.

This raises the following question:

Question 11. Can every continuous function be written as the sum of two functions which satisfy Lusin's condition (N)?

We conclude this survey with a question which is of interest in its analogy to the decomposition of a function of bounded variation into the sum of two monotone functions:

Question 12. Can any continuous function which is BVG be written as the sum of two continuous generalized monotone functions?

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