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# Limit of Simply Continuous Function

Let $X$ be a topological space and let $(Y, d)$ be a metric space. For a subset $A$ of a topological space let $C\ell A$ and Int $A$ denote the closure and interior of $A$, respectively. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of natural, rational and real numbers, respectively. If $\mathcal{F} \subset S^Y$ is a class of functions defined on $X$ with values in $Y$, we denote by $U(\mathcal{F}), D(\mathcal{F})$ and $P(\mathcal{F})$ the collection of all uniform, quasiuniform and pointwise limits of sequences taken from $\mathcal{F}$, respectively.

Recall that a sequence $(f_n)$, $f_n : X \to Y$, converges quasiuniformly to $f : X \to Y$ (See [13], page 143.) if it converges pointwise to $f$ and $\forall \varepsilon > 0\ \forall m \in \mathbb{N}\ \exists\, p \in \mathbb{N}\ \forall x \in X : \ \min\{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+p}(x), f(x))\} < \varepsilon$. Evidently $U(\mathcal{F}) \subset D(\mathcal{F}) \subset P(\mathcal{F})$.

The aim of this paper is to investigate the sets $U(\mathcal{F})$, $D(\mathcal{F})$ and $P(\mathcal{F})$ for the class of simply continuous functions. We recall that a function $f : X \to Y$ is simply continuous (See [1].) if $f^{-1}(V)$ is a simply open set in $X$ for each open set $V$ in $Y$. A set $A$ is simply open if it is the union of an open set and a nowhere dense set. A function $f : X \to Y$ is cliquish at a point $x \in X$ (See [11].) if for each $\varepsilon > 0$ and each neighborhood $U$ of $x$ there is a nonempty open set $G \subset U$ such that $d(f(y), f(z)) < \varepsilon$ for each $y, z \in G$. A function $f : X \to Y$ is said to be cliquish if it is cliquish at each point $x \in X$. A funciton $f : X \to Y$ is quasicontinuous at a point $x \in X$ (See [11].) if for each neighborhood $U$ of $x$ and each neighboorhood $V$ of $f(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Denote by $Q_f$ the set of all points at which $f$ is quasicontinuous. If $Q_f = X$, then $f$ is said to be quasicontinuous.

Denote by $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{B}$ the set of all functions which are quasicontinuous, simply continuous, cliquish and have the Baire property (with $X$ as the domain and $Y$ as the range), respectively. Evidently $\mathcal{Q} \subset \mathcal{S} \subset \mathcal{B}$ and $\mathcal{Q} \subset \mathcal{K} \subset \mathcal{B}$. In [12] it is shown that if $X$ is a Baire space and $Y$ is a separable metric space, then $\mathcal{S} \subset \mathcal{K}$. Example 1 in [5] shows that these assumptions cannot be omitted. It is shown in [11] that $U(\mathcal{K}) = \mathcal{K}$ and that $P(\mathcal{B}) = \mathcal{B}$. If $X$ is a Baire space, then $D(\mathcal{K}) = \mathcal{K}$. (See [7].) Proposition 1 in [6] shows that this is not true for arbitrary $X$. In [8] it is shown that $P(\mathcal{K}) = \mathcal{K}$ for $X = Y = \mathbb{R}$ and in [9] for

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$X = \mathbb{R}^m$ and $Y = \mathbb{R}$. We shall show that this assertion holds for an arbitrary topological space $X$ and a separable metric space $Y$.

**Lemma 1** *Let $f : Y \to Y$ be such that the set $X \setminus Q_f$ is nowhere dense. Then $f$ is simply continous.*

*Proof.* Let $V$ be an open set in $Y$. Then by [4] $Q_f \cap (f^{-1}(V) \setminus \mathrm{Int}\, f^{-1}(V))$ is nowhere denseand hence the set $f^{-1}(V) \setminus \mathrm{Int}\, f^{-1}(V)) \subset ((f^{-1}(V) \setminus \mathrm{Int}\, f^{-1}(V)) \cap Q_f) \cup (X \setminus Q_f)$ is nowhere dense. Therefore $f$ is simply continuous.

**Theorem 1** *Let $X$ be a topological space and let $(Y, d)$ be a separable metric space. Then $P(\mathcal{S}) = \mathcal{B}$.*

*Proof.* Let $f \in \mathcal{B}$. By [10] there are disjoint open sets $C$ and $D$ such that $C$is a Baire space, $D$ is of the first category and $C \cup D$ is dense in $X$. Then $D = \bigcup_{i=1}^{\infty} D_i$, where each $D_i$ is a nowhere dense set and $D_i \subset D_{l+1}$ for each $i \in \mathbb{N}$. Since $f$ has the Baire property, there is a set $A$ of the first category such that $f|_{X \setminus A}$ is continuous. Then $C \cap A = \bigcup_{i=1}^{\infty} A_i$, where each $A_i$ is a nowhere dense set and $A_i \subset A_{i+1}$ for each $i \in \mathbb{N}$. Set $g = f|_{X \setminus A}$.

Let $n \in \mathbb{N}$. Since $Y$ is separable, $Y = \bigcup_{j=1}^{\infty} S(u_j^n, \frac{1}{n})$, where $\{u_j^n : j \in \mathbb{N}\}$ is a countable dense set in $Y$. ($S(u, \varepsilon)$ is the open sphere of radius $\varepsilon > 0$ about $u$.) Since $g$ is continuous, for each $k \in \mathbb{N}$ there is an open set $T_j^n$ in $C$ such that $g^{-1}(S(u_j^n, \frac{1}{n})) = T_j^n \setminus A$. Put $W_1^n = T_1^n$ and $W_j^n = T_j^n \setminus \bigcup_{i=1}^{j-1} T_j^n$ for $j > 1$ and $B_j^n = \mathrm{Int}\, W_j^n$ for each $j \in \mathbb{N}$. Since each $T_j^n$ is open, each $W_j^n$ is simply open and hence each $K_j^n = W_j^n \setminus V_j^n$ is nowhere dense. Evidently the sets $V_j^n$ are pairwise disjoint. Set $W^n = \bigcup_{j=1}^{\infty} W_j^n$, $V^n = \bigcup_{j=1}^{\infty} V_j^n$ and $K^n = \bigcup_{j=1}^{\infty} K_j^n$.

If $x \in C \setminus A$, then there is $u \in \mathbb{N}$ such that $x \in g_{-1}(S(u_j^n, \frac{1}{n}))$ and hence $x \in T_j^n$. Therefore $C \setminus A \subset W^n$ and hence $W^n$ is dense in $C$. Since $W^n = \bigcup_{j=1}^{\infty} T_j^n$, the set $W^n$ is open the set $V^n$ is also open and hence the set $K^n$ is simply open. However the set $K^n$ is of the first category and hence Int $K^n$ is the empty set; that is, $K^n$ is nowhere dense. This yields that $V^n$ is dense in $C$.

Now define a sequence of functions $f_n : X \to Y$ as follows:

$$f_n(x) = \begin{cases} u_j^n & \text{if } x \in B_j^n \setminus C\ell\, A_n \\ u_1^n & \text{if } x \in D \setminus C\ell\, D_n \\ f(x) & \text{otherwise.} \end{cases}$$

The set $F = (X \setminus (C \cup D)) \cup C\ell\, D_n \cup C\ell\, A_n \cup (C \setminus V^n)$ is nowhere dense and $f_n$ is continuous on $x \setminus F$. Hence by 1, $f_n$ is simply continuous. It is easy to see that the sequence $(f_n)$ converges to $f$. Thus $\mathcal{B} \subset P(\mathcal{S})$.

Evidently $P(\mathcal{S}) \subset P(\mathcal{B}) = \mathcal{B}$.

By [3] we have $U(\mathcal{S}) \neq \mathcal{S}$. In fact the following assertion is true.

**Theorem 2** *Let $X$ be a Baire space and let $(Y, d)$ be a separable metric space. Then $D(\mathcal{S}) = U(\mathcal{S}) = \mathcal{K}$.*

*Proof.* Let $f \in \mathcal{K}$ and let $n \in \mathbb{N}$. Then there is a countable dense set $\{u_j^n : j \in \mathbb{N}\}$ in $Y$ such that $Y = \cup j = 1^\infty S(u_j^n, \frac{1}{n})$. For $j \in \mathbb{N}$ put $T_j^n = \text{Int } f^{-1}(S(u_j^n, \frac{1}{n}))$, $W_j^n = T_j^n \setminus \cup_{i=1}^{j-1} T_i$ and $V_j^n = \text{Int } W_j^n$. If $x \in C_f$ (Where $C_f$ is the set of all points of continuity of $f$.), then there is $j \in \mathbb{N}$ such that $f(x) \in S(u_j^n, \frac{1}{n})$. The continuity of $f$ at $x$ gives $x \in T_j^n$. Since $X$ is a Baire space, the set $C_f$ is dense in $X$. (See [7].) Similarly as in 1 we can show that the set $\cup_{j=1}^\infty V_j^n$ is dense in $X$.

Let

$$f_n(x) = \begin{cases} u_j^n & \text{if } x \in V_j^n \\ f(x) & \text{otherwise.} \end{cases}$$

Then $f_n$ is simply continuous by 1. Since for each $x \in X$ we have $d(f_n(x), f(x)) < \frac{1}{n}$, the sequence $(f_n)$ converges uniformly to $f$. Therefore $\mathcal{K} \subset U(\mathcal{S})$.

According to [12] and [7] we have $U(\mathcal{S}) \subset D(\mathcal{S} \subset D(\mathcal{S}) = \mathcal{K}$.

**Theorem 3** *Let $X$ be a Baire space and let $(Y, d)$ be a separable metric space. Then $\mathcal{K} = U(\mathcal{K}) = D(\mathcal{K}) = U(\mathcal{S}) = D(\mathcal{S}) \subset \mathcal{B} = P(\mathcal{S}) = P(\mathcal{K}) = P(\mathcal{B}) = D(\mathcal{B}) = U(\mathcal{B})$.*

By [11] we have $U(\mathcal{Q}) = \mathcal{Q}$. In [8] and [9] it is shown that $P(\mathcal{Q}) = \mathcal{K}$ for $X = \mathbb{R}^m$ and $Y = \mathbb{R}$. If $X = Y = \mathbb{R}$, then by [12] $D(\mathcal{Q}) = \mathcal{K}$. Hence we have the following.

**Theorem 4** *Let $X = Y = \mathbb{R}$. Then $\mathcal{Q} = U(\mathcal{Q}) \subset \mathcal{S} \subset \mathcal{K} = U(\mathcal{K}) = D(\mathcal{K}) = U(\mathcal{S}) = D(\mathcal{S}) = D(\mathcal{Q}) = P(\mathcal{Q}) =\subset \mathcal{B} = P(\mathcal{B}) = D(\mathcal{B}) = U(\mathcal{B}) = P(\mathcal{K}) = P(\mathcal{S})$.*

We will now show another manner in which functions having the Baire property can be approximated by simply continuous functions.

**Theorem 5** *Let $(Y, d)$ be a locally compact separable metric space. Then $f : X \to Y$ has the Baire property if and only if $f(x) \neq g(x)\}$ is of the first category.*

*Proof.* Let $f \in \mathcal{B}$. First let us assume that $X$ is a Baire space. Then there is a residual set $A$ such that $f|_A$ is continuous. Set

$$C(f, x, A) = \cap_{U \in \mathcal{U}_x} C\ell\, f(A \cap U)$$

(Where $\mathcal{U}_x$ is the family of all neighborhoods of $x$.) and

$$E = \{x \in X : C(f, x, A) = \emptyset\}.$$

Let $x \in A$. Since $Y$ is locally compact, there is a closed compact neighborhood $W$ of $f(x)$. Then there is an open neighborhood $U_x$ of $x$ such that $f|_A(U_x) = f(A \cap U_x) \subset W$. Then $C\ell\, f(A \cap U_x) \subset W$. Let $u \in U_x$. Then $(C\ell\, f(A \cap U \cap U_x))_{U \in \mathcal{U}_u}$ is a family of closed subset of $W$ with the finite intersection property. Hence $\cap_{U \in \mathcal{U}_u}(C\ell\ f(A \cap U \cap U_x \cap U) \neq \emptyset$ and therefore $C(f, u, A) \neq \emptyset$. This yields $U_x \cap E = \emptyset$. therefore

(1) $$A \cap C\ell\, E = \emptyset.$$

Since $A$ is dense, $E$ is nowhere dense. For $x \in X \setminus E$ choose $x^* \in C(f, x, A)$ and define $g : X \to Y$ as

$$g(x) = \begin{cases} f(x) & \text{if } x \in E \\ x^* & \text{otherwise.} \end{cases}$$

Evidently $\{x \in X : f(x) \neq g(x)\}$ is of the first category. We will show that $g$ is simply continuous.

Let $x \in A$ and let $T$ be a neighborhood of $g(x)$. Let $V$ be a neighborhood of $g(x)$ such that $C\ell\ V \subset T$. Then there is an open neighborhood $H$ of $x$ such that $f(A \cap H) \subset V$. Then according to (1) $U = H \setminus C\ell\ E$ is an open neighborhood of $x$. Let $u \in U$. Then $u \notin E$ and hence $g(u) \in C(f, u, A)$. Thus $g(u) \in C\ell\, f(A \cap U) \subset C\ell V \subset T$. This yields

(2) $$A \subset C_G.$$

Now let $x \in X \setminus E$. Let $U$ be a neighborhood of $x$ and let $W$ be an open neighborhood of $f(x)$. Then $f(U \cap A) \cap W \neq \emptyset$. Let $t \in f(U \cap A) \cap W$. Then there is a $y \in U \cap A$ such that $f(y) = t$. By (2), $y \in C_g$ and $f(y) = g(y)$. Hence there is an open set $G$ such that $U \in G \subset Y$ and $f(G) \subset W$; that is, $x \in Q_g$. Therefore $X \setminus E \subset Q_g$ and the set $X \setminus Q_g$ is nowhere dense. By 1, $g$ is simply continuous.

If $X$ is an arbitrary topological space, then by [10], there are disjoint open sets $C$ and $D$ such that $C$ is a Baire space, $D$ is of the first category and $C \cup D$ is dense in $X$. Let $a \in Y$ and let $h : C \to Y$ be a simply continuous function such that $\{x \in C : h(x) \neq f(x)\}$ is of the first category. Then the function $g : X \to Y$ defined by

$$g(x) = \begin{cases} h(x) & \text{if } x \in C \\ a & \text{otherwise,} \end{cases}$$

is a simply continuous function such that $\{x \in X : g(x) \neq f(x)\}$ is of the first category. On the other hand, if $g : X \to Y$ is a simply continuous function and $f : X \to Y$ is a function such that $\{x \in X : f(x) \neq g(x)\}$ is of the first category, then $f$ has the Baire property.

The following example shows that the assumption "$Y$ is locally compact" in 5 cannot be omitted.

**Example 1** Let $X = \mathbb{R}$ (with the usual metric) and let $Y = \mathbb{R}$ with the (separable) metric $d$ :

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in \mathbb{R}\setminus\mathbb{Q} \\ 0 & \text{if } x = y \\ \max\{1, |x-y|\} & \text{otherwise.} \end{cases}$$

Let $f : X \to Y$, $f(x) = x$ for each $x \in X$. Then $f$ has the Baire property. Let $g : X \to Y$ be a simply continuous function such that $\{x \in X : f(x) \neq g(x)\}$ is of the first category. We will show that $g$ cannot be simply continuous.

If $g(x) \in \mathbb{Q}$ for each $x \in \mathbb{Q}$, then since $\mathbb{Q}$ is open in $Y$, $g^{-1}(\mathbb{Q})$ must be a dense set of the first category in $X$ and hence it is not simply open. So suppose $g(x) \in \mathbb{R}\setminus\mathbb{Q}$ for some $x \in \mathbb{Q}$. If $V$ is an open neighborhood of $g(x)$ and $U$ is a "small" neighborhood of $x$, then $g^{-1}(V)$ is dense in $U$ and hence by [4] Int $g^{-1}(V) \cap U \neq \emptyset$. This yields $x \in Q_g$. However, if $g(x) \in \mathbb{R}\setminus\mathbb{Q}$, then for $\alpha = \frac{|g(x)-x|}{2} > 0$ we have $g(S(x,\alpha)\setminus\mathbb{Q}) \cap S(g(x),\alpha) = \emptyset$; that is, $x \notin Q_g$.

**Remark 1** *If $Y$ is a compact separable metric space, thenthe function $g$ from 5 is quasicontinuous.*

**Remark 2** *From the proof of 5 it follows that the function $g$ is such that $X\setminus Q_g$ is nowhere dense. This is stronger than simple continuity. (The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = r(x) + x$, where $r$ is the Riemann function, is simply continuous by [3]. However $\mathbb{R}\setminus Q_f$ is dense in $\mathbb{R}$.) From the proofs of 1 and 2 it follows that a function with the Baire property (cliquish function) is the pointwise (uniform) limit of functions $f_n$ such that $X\setminus C_{f_n}$ are nowhere dense sets. This is not true for 5. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sum_{n:q_n<x} 2^{-n}$ (Where $\mathbb{Q} = \{q_1, q_2, \cdots\}$ is a one-to-one sequence.) is quasicontinusous. However for each function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R}\setminus C_g$ is nowhere dense, the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ contains a nonempty open set.*

**Remark 3** *Applying a well-known theorem due to Blumberg (See for example [10], page 30.) to the proof of 5 we get the following assetion. Let $X$ be a Baire metric space and let $f : X \to \mathbb{R}$ be an arbitrary (locally bounded) function. Then there is a simply continuous (quasicontinuous) function $g : X \to \mathbb{R}$ such that $\{x \in X : f(x) = g(x)\}$ is dense in $X$.*

# References

[1] Biswas, N., *On some mappings in topological spaces*, Bull Calcutta Math. Soc 61 (1969) 127–135

[2] Borsík, J., *Quasiuniform limits of quasicontinuous functions*, Math Slovaca, (to appear)

[3] Borsík, J. and Doboš, J., *A note on real cliquish functions*, Real Anal. Exchange, 18 (1992–93), this issue.

[4] Borsík, J. and Doboš, J., *On simple continuity points*, Real Anal. Exchange, 16 (1990–91), 552–558.

[5] Doboš, J., *Simple continuity and cliquishness*, Časopis Pěst mat., 112 (1987), 355–358.

[6] Doboš, J., *Some generalization of the notion of continuity and quasi-uniform convergence*, Časopis Pěst mat., 106 (1981), 431–434.

[7] Doboš, J. and Šalát, T., *Cliquish functions, Riemann integrable functions and quasi-uniform convergence*, Acta Math. Univ Comen., 40–41 (1982), 219–223.

[8] Grande, Z., *Sur la quasi-continuité et la quasi-coninuité approximative,*, Fund. Math. 129 (1988), 167–172.

[9] Grande, Z., Natkaniec, T. and Stronska, E., *Lattices, algebras and Baire's systems generated by some families of functions*, Real Anal. Exchange, 13 (1987–88), 61–66.

[10] Haworth, R. C. and McCoy, R. A., *Baire spaces*, Dissertationes Math., 141, Warszawa, 1977.

[11] Marcus, S., *Sur les fonctions quasicontinues au sens de S. Kempisty*, Colloq. Math. 8 (1961), 47–53.

[12] Neubrunnová, A., *On certain generalizations of the notion of continuity*, Mat. časopis, 23 (1973), 374–380.

[13] Sikorski, R., *Real Functions I* (in polish), PWN, Warszawa, 1958.