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# On Some Questions of R. Gordon Related to Approximate and Dyadic Henstock Integrals

In [1] Russell A. Gordon gave new descriptive characterizations of the approximate Henstock integral and of the dyadic Henstock integral using $ACG_s$ and $ACG_d$ functions and posed three questions related to these classes of functions.

Here we give answers to these questions and make some other related remarks.

For the definitions and notation the reader is referred to [1].

**1.** "Is every Denjoy-Khintchine integrable function $S$-Henstock integrable? This amounts to asking whether or not an $ACG$ function is $ACG_s$."

The answer is "No!" I.e., there exists an $ACG$ function which is not $ACG_s$. To prove this we can use an example constructed by Tolstov [2].

Let $\rho_k^n = (a_k^n, b_k^n)$, $k = 1, 2, \ldots, 2^{n-1}$, $n = 1, 2, \ldots$, be contiguous intervals of the Cantor ternary set $C$, $|\rho_k^n| = 3^{-n}$. Let $c_k^n, d_k^n$ be such that $a_k^n < c_k^n < d_k^n < b_k^n$ and $c_k^n - a_k^n = b_k^n - d_k^n = 0(|b_k^n - a_k^n|)$ as $n \to \infty$. Put

$$F(x) = \begin{cases} n^{-1} & \text{for } x \in (c_k^n, d_k^n), \\ 0 & \text{for } x \in C, \\ \text{is linear} & \text{for } x \in [a_k^n, c_k^n] \cup [d_k^n, b_k^n]. \end{cases}$$

Obviously $F(x)$ is an $ACG$ function.

It is proved in [2] that $F$ is not an indefinite approximate Perron integral. But the $AP$-integral is equivalent to the $S$-Henstock type integral (See [3], where Th.1.6.1 relates to a general Henstock type integral, where the $S$-integral is a special case of it). According to [1] $ACG_s$ is a descriptive characterization of the $S$-Henstock integral. So, $F$ is not an $ACG_s$ function.

**2.** "Is there a continuous function that has a dyadic derivative at each point, but is not differentiable on an uncountable set?"

The positive answer is given here by constructing an example of a function with the required properties.

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On the unit interval define a Cantor type set $P$ by deleting a sequence of open intervals. Let $E_1^1$ be the open middle half of the unit interval $[0,1]$; i.e., $E_1^1 = [(\frac{1}{4}, \frac{3}{4})$. Let $E_1^2$ and $E_2^2$ be the open middle halves of the two closed intervals of $[0,1]\setminus E_1^1$; i.e., $E_1^2 = (\frac{1}{16}, \frac{3}{16})$, $E_2^2 = (\frac{13}{16}, \frac{15}{16})$. Delete these and let $E_1^3, E_2^3, E_3^3, E_4^3$ be the open middle halves of the remaining four closed intervals, etc. The perfect set is defined as

$$P = [0,1]\setminus \bigcup_{n,k} E_k^n.$$

Notice that each interval $E_k^n$ is the union of two dyadic intervals of order $2n$.

Put

$$F(x) = \begin{cases} \frac{1}{n}\sin^2 2\pi \dfrac{x-a}{b-a} & \text{for } x \in E_k^n = (a,b) \\ 0 & \text{for } x \in P. \end{cases}$$

$F(x)$ is continuous on $[0,1]$, differentiable on each $E_k^n$ and at the end points of $E_k^n$ from inside but is not differentiable on $P$. (For each $x \in P$ there exists a sequence $\{x_n'\}$, $x_n' \to x$, such that

$$\left|\frac{F(x_n')}{x_n' - x}\right| > \frac{2^{2n}}{n}.)$$

Let $D$ represent the set of dyadic rational numbers in $[0,1]$.

$F_d'(x) = 0$ at each $x \in P\setminus D$ because for such $x$ and for all $n$, $F(x_n + 2^{-n}) - F(x_n) = 0$, (We are using the notation from [1].) and $F_d'(r) = 0$ at each $r \in P \cup D$ because such $r$ is an end point of some $E_k^n$ and

$$F(r + 2^{-n}) - F(r) = F(r - 2^{-n}) - F(r) = 0$$

for all $n$ and for all points $r + 2^{-n}$, $r - 2^{-n}$ outside of $E_k^n$; the derivative from inside of $E_k^n$ being equal to 0.

**Remark 1.** In fact a stronger result holds. There exists a continuous function that has a dyadic derivative everywhere, but is not differentiable on a set of positive measure. At the same time the following theorem was proved in [4. Lemma 4 and Theorem 2]: If $F$ is dyadicly differentiable on a measurable set $E$, then $F$ approximately differentiable almost everywhere on $E$ and

$$F_{ap}'(x) = F_d'(x) \text{ a.e.}$$

**3.** Does an $ACG_d$ function have a dyadic derivative almost everywhere?

To get the positive answer we need the following theorem: $F_d'(x)$ *exists at almost every point of a set where either*

$$\overline{F}_d'(x) < +\infty \textit{ or } \underline{F}_d'(x) > -\infty.$$

(This is a special case of the theorem from Saks [5, Ch. VI. p 1921.)

Having this, repeat the arguments of Theorem 21 of [1] to get the required result.

**Remark 2.** We note in conclusion that in contrast to the case of an $ACG_s$ function (and in particular of $ACG_\delta$ function [6]), $ACG_d$ functions can fail to satisfy Lusin's condition $(N)$. (See [7].)

## References


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