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Products of Darboux Functions

Let us establish some terminology to be used. R denotes the set of all reals. I denotes a non-degenerate closed interval. If A is a planar set, we denote its x-projection by dom(A) and y-projection by rng(A).

We shall consider real functions defined on a real interval. No distinction is made between a function and its graph. The symbols $C^-(f,x)$ and $C^+(f,x)$ denote the left and right cluster sets of f at the point x. The symbol $C(f)$ denotes the set of all continuity points of f. The notation $[f>0]$ means the set $\{x : f(x) > 0\}$. Likewise for $[f = 0]$, $[f \neq 0]$, etc. For subsets $A, B \subset R$ let $\mathcal{D}^*(A, B)$ denote the class of all functions $f : A \longrightarrow B$ such that $\mathrm{cl}_A f^{-1}(y) = A$ for each $y \in B$. Let us remark that if A is an F_σ set and A is c dense-in-itself, then the class $\mathcal{D}^*(A, R)$ contains Baire 2 functions (see [2]).

The function f is said to be Darboux if $f(C)$ is connected whenever C is a connected subset of the domain of f. If each open set containing f also contains a continuous function with the same domain as f, then f is almost continuous [2]. It is clear that if $f : I \longrightarrow R$ is almost continuous, then f is connected and, therefore, it has the Darboux property. Moreover, if f meets each closed subset F of $I \times R$ with $\mathrm{int}(\mathrm{dom}(F)) \neq \emptyset$, then f is almost continuous [2].

We shall use the following set-theoretical assumption.

$A(c)$ – the union of less than 2^ω many first category subsets of R is of the first category again.

Note that this statement is a consequence of Martin's Axiom and therefore also the Continuum Hypothesis (see e.g. [2]).

It is well known that each real-valued function defined on a real interval can be expressed as a sum of two Darboux functions [2]. This fact was improved by Fast in the following way: if $\mathcal{F}$ is a collection of c-many real functions then there exists a function g such that $f+g$ is Darboux for each $f \in \mathcal{F}$ [2]. In 1967, Mišik proved that for each countable family $\mathcal{F}$ of Baire α functions (where $\alpha > 1$) there exists a Baire α function g such that $f + g$ has the Darboux property for every $f \in \mathcal{F}$ [2]. In 1984, Pu and Pu proved the analogous result for finite

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families of Baire 1 functions [2]. In 1974, Kellum proved that Fast's theorem holds if "*Darboux*" is replaced by "*almost continuity*". In the present paper we state some similar results with respect to products of functions.

First, let us remark that a general function may not be a product of Darboux functions (and therefore also almost continuous functions) [2], see for example the function f given by $f(x) = 1$ for $x \neq 0$ and $f(0) = -1$. Products of two Darboux functions, Darboux Baire 1 functions and almost continuous functions are characterized in [2], [2] and [2], respectively.

Theorem 1 *If $\mathcal{F}$ is a countable family of Baire α functions and $\alpha > 1$, then there exists a non-zero Darboux Baire α function g such that:*

1. *fg has the Darboux property for each $f \in \mathcal{F}$,*
2. *the set $[g \neq 0]$ has Lebesgue measure zero and is of the first category (hence g and all fg, for $f \in \mathcal{F}$, are measurable and have the Baire property).*

Proof. Let us assume that $\mathcal{F} = \{f_n : n \in N\}$ is a given family of Baire α functions defined on I. For $n \in N$, let G_n be the union of all subintervals J of I such that $J \cap [f_n \neq 0]$ has the cardinality less than 2^ω (in fact, since f_n is Borel measurable, this set must be countable). Let us define $H_n = \overline{G_n} \cap [f_n \neq 0]$. Since each H_n is of the first category, $A = \bigcup_{n \in N} H_n$ is of the first category too. Let $(I_n, t_n)_{n \in N}$ be a sequence of all sets of the form $J \times \{n\}$, where J is a subinterval of I with rational end-points and $n \in N$. Inductively we can choose a sequence $(C_n)_{n \in N}$ of Cantor sets (having measure zero) such that for each $n \in N$ the following conditions are fulfilled.

(i) If $I_n \subset \overline{G_{t_n}}$, then $C_n \subset I_n \cap [f_{t_n} = 0] \setminus (A \cup \bigcup_{i<n} C_i)$.

(ii) If $I_n \setminus \overline{G_{t_n}} \neq \emptyset$, then $C_n \subset I_n \cap [f_{t_n} \neq 0] \setminus (A \cup \bigcup_{i<n} C_i)$.

For any $n \in N$, let us put $T_n = \bigcup\{C_i : t_i = n\}$. Then each T_n is of the type F_σ and c-dense in I. Moreover, all T_n have measure zero, are of the first category and satisfy the following conditions.

(iii) $T_n \subset I \setminus A$,

(iv) $T_n \setminus \overline{G_n} \subset [f_n \neq 0]$.

Now for any $n \in N$, let $T_{1,n}, T_{2,n}$ be two disjoint F_σ subsets of the set $T_n \setminus \overline{G_n}$ both c-dense in $I \setminus \overline{G_n}$. Let us fix Baire 2 functions $g_{0,n} \in \mathcal{D}^*(G_n \cap T_n, R)$, $g_{1,n} \in \mathcal{D}^*(T_{1,n}, R)$ and $h_n \in \mathcal{D}^*(T_{2,n}, R)$.
We define $g : I \longrightarrow R$ as follows:

$$g(x) = \begin{cases} g_{0,n}(x) & \text{for } x \in G_n \cap T_n, \\ g_{1,n}(x) & \text{for } x \in T_{1,n}, \\ h_n(x)/f_n(x) & \text{for } x \in T_{2,n}, \\ 0 & \text{otherwise.} \end{cases}$$

We shall verify that g fulfills the conditions (1) and (2). First, let us fix a subinterval $J \subset I$. If $J \subset \overline{G_0}$, then $g(J) \supset g_{0,0}(J \cap G_0 \cap T_0) = R$. Otherwise, $J \setminus \overline{G_0} \neq \emptyset$ and $g(J) \supset g_{1,0}(J \cap T_{1,0}) = R$. Thus $g \in \mathcal{D}^*(I, R)$ and consequently, $g \not\equiv 0$. Now fix $n \in N$ and a subinterval J of I. If $J \subset \overline{G_n}$, then $(f_n g)|J \equiv 0$. If $J \subset I \setminus \overline{G_n}$, then $(f_n g)|J \supset h_n|(J \cap T_{2,n})$ and hence $(f_n g)|J \in \mathcal{D}^*(J, R)$. Additionally, let us remark that $f_n g|(\overline{G_n} \setminus G_n) \equiv 0$. These three conditions imply easily Darboux property of $f_n g$.
Since $[g \neq 0] \subset \bigcup_{n \in N} T_n$, the function g equals zero except a first category set of Lebesgue measure zero.
Finally, it is easy to verify that g is a Baire α function. □

Theorem 2 *Let us assume $A(c)$. If $\mathcal{F}$ is a family of Baire 1 functions of the power less than 2^ω, then there exists a non-zero function $g \in \mathcal{DB}_1$ such that fg has the Darboux property for each $f \in \mathcal{F}$.*

Proof. Let us assume that $\kappa < 2^\omega$ and $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ is a family of Baire 1 functions. Then for each $\alpha < \kappa$ the set $C(f_\alpha)$ of all continuity points of f_α is residual and, by $A(c)$, $B = \bigcap_{\alpha<\kappa} C(f_\alpha)$ is residual too. We choose a Cantor set $C \subset B$ and a non-zero function $g \in \mathcal{DB}_1$ such that $[g \neq 0] \subset C$ (see e.g. [2], p.13). Note that for a given $\alpha < \kappa$ the product $f_\alpha g$ is a Baire 1 function. Using the Young's criterion (see [2], p. 9), we shall verify that $f_\alpha g$ has the Darboux property. Fix $x_0 \in I$. We consider two cases.

(a) $x_0 \in C(f_\alpha)$. Since $g \in \mathcal{DB}_1$, there exist sequences $x_n \nearrow x_0$, $y_n \searrow x_0$ with $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(y_n) = g(x_0)$. Then $\lim_{n\to\infty}(f_\alpha g)(x_n) = \lim_{n\to\infty}(f_\alpha g)(y_n) = (f_\alpha g)(x_0)$.

(b) $x_0 \notin C(f_\alpha)$. Then $g(x_0) = 0$, and since $[g = 0]$ is dense in I, we can select two sequences $x_n \nearrow x_0$, $y_n \searrow x_0$ with $g(x_n) = g(y_n) = 0$ for $n \in N$. Thus $\lim_{n\to\infty}(f_\alpha g)(x_n) = \lim_{n\to\infty}(f_\alpha g)(y_n) = 0 = (f_\alpha g)(x_0)$.

Consequently, $f_\alpha g$ has the Darboux property. □

Theorem 3 *Let us assume $A(c)$. If $\mathcal{F}$ is a family of real functions of the power less than 2^ω, then there exists an almost continuous function $g \not\equiv 0$ such that fg is almost continuous for each $f \in \mathcal{F}$.*

Proof. Let us assume that $\kappa < 2^\omega$ and $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$. For $\alpha < \kappa$, let G_α be the union of all subintervals J of I such that $J \cap [f_\alpha \neq 0]$ is of the first category. Then $A = \bigcup_{\alpha<\kappa}((\overline{G_\alpha} \setminus G_\alpha) \cup (G_\alpha \cap [f_\alpha \neq 0]))$ is of the first category again. Let $(F_\alpha)_{\alpha<2^\omega}$ be a sequence of all closed subset of $I \times R$ with $\mathrm{int}(\mathrm{dom}(F_\alpha)) \neq \emptyset$. Let $\varphi : \kappa \times 2^\omega \longrightarrow 2^\omega$ be a bijection. For each $\gamma < 2^\omega$ such that $\gamma = \varphi(\alpha, \beta)$, we choose $z_\gamma, t_\gamma \in R^2$ such that:

(*i*) if $\text{int}(\text{dom}(F_\beta)) \setminus \overline{G_\alpha} \neq \emptyset$, then $z_\gamma, t_\gamma \in F_\beta$,

$$\begin{array}{rcl} \text{dom}(z_\gamma) & \in & [f_\alpha \neq 0] \cap \text{int}(\text{dom}(F_\beta)) \setminus (\overline{G_\alpha} \cup A \cup \text{dom}\{z_\nu, t_\nu : \nu < \gamma\}), \\ \text{dom}(t_\gamma) & \in & [f_\alpha \neq 0] \cap \text{int}(\text{dom}(F_\beta)) \\ & \setminus & (\overline{G_\alpha} \cup A \cup \text{dom}\{z_\mu, t_\nu : \nu < \gamma, \mu \leq \gamma\} \cup \{0\}), \end{array}$$

(*ii*) if $\text{int}(\text{dom}(F_\beta)) \subset \overline{G_\alpha}$, then $t_\gamma = (0,0)$, $z_\gamma \in F_\beta$ and

$$\text{dom}(z_\gamma) \in G_\alpha \cap [f_\alpha = 0] \setminus \text{dom}\{z_\nu, t_\nu : \nu < \gamma\},$$

Now we define the function $g : I \longrightarrow R$ by

$$g(x) = \begin{cases} \text{rng}(z_\gamma) & \text{if } x = \text{dom}(z_\gamma),\ \gamma < 2^\omega \\ \text{rng}(t_\gamma)/f_\alpha(x) & \text{if } x \neq 0,\ x = \text{dom}(t_\gamma),\ \varphi(\gamma) = (\alpha, \beta),\ \alpha, \beta, \gamma < 2^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Since g meets every blocking set, g is almost continuous [2]. Moreover, it is easy to observe that $g \in \mathcal{D}^*(I, R)$ and therefore, $g \not\equiv 0$. Finally we will verify that for $\alpha < \kappa$ the function $f_\alpha g$ is almost continuous. Notice that $(f_\alpha g)|J \equiv 0$ for each component J of the set $\text{int}(\overline{G_\alpha})$. If J is a component of $I \setminus \overline{G_\alpha}$, then $(f_\alpha g)|J$ is almost continuous and, moreover, $(f_\alpha g)|J \in \mathcal{D}^*(J, R)$. Finally we observe that the set $C = \overline{G_\alpha} \setminus G_\alpha$ is closed and nowhere dense. Thus $f_\alpha g$ fulfills the following conditions:

(*a*) $f_\alpha g(x) = 0$ for $x \in C$,

(*b*) $0 \in C^-((f_\alpha g)|(I \setminus C), x) \cap C^+((f_\alpha g)|(I \setminus C), x)$ for $x \in C$,

(*c*) $(f_\alpha g)|J$ is almost continuous for any component J of the set $I \setminus C$.

By Lemma 3 [2], we conclude that the function $f_\alpha g$ is almost continuous. □

Note that the following example shows that Theorems 2 and 3 can not be improved for families $\mathcal{F}$ with $card(\mathcal{F}) = 2^\omega$.

Example 1 *There exists a family $\mathcal{F}$ of 2^ω-many Baire 1 functions such that for every non-zero function g there is some $f \in \mathcal{F}$ such that fg does not have the Darboux property.*

Indeed, let $h = 1$ and $\mathcal{F}_0$ be the family of all characteristic functions χ_x of singletons, and let $\mathcal{F} = \{h\} \cup \mathcal{F}_0$. Let us assume that $g : I \longrightarrow R$ is a function such that fg has the Darboux property for each $f \in \mathcal{F}$. Since $g = gh$, g has the Darboux property. Let us fix $x \in I$. Since $(\chi_x g)(y) = g(x)\chi_x(y) = 0$ for all $y \neq x$ and $\chi_x g$ has the Darboux property, $g \equiv 0$.

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