Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 221–224

Zoltán Buczolich*, Eötvös Loránd University, Department of Analysis, Budapest, Múzeum krt 6-8, H-1088, Hungary.

The n-Dimensional Gradient Has the 1-Dimensional Denjoy-Clarkson Property

In this paper we present a partial result related to the gradient problem of C.E.Weil [Q]. The original problem is the following one. "Assume that f is a differentiable real valued function of n real variables and let $g = \nabla f$ denote its gradient, which is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let G be a nonempty open subset of $\mathbb{R}^n$. Is it true that $g^{-1}(G)$ is either empty or has positive n-dimensional measure?" Though we do not answer this question in this paper we shall show a similar result, namely, $g^{-1}(G) = (\nabla f)^{-1}(G)$ is either empty or has positive 1-measure in $\mathbb{R}^n$ in the sense of Hausdorff measures (cf., [R] Chapter 2, or [F] Chapter 1, Section 1.2).

For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ put $||x|| = \sqrt{\sum_{i=1}^n x_i^2}$. For an $x \in \mathbb{R}^n$ we denote the open ball centered at x and of radius r by $B(x,r)$, that is, $B(x,r) = \{y : ||x - y|| < r\}$. The boundary of $B(x,r)$ will be denoted by $C(x,r)$, that is, $C(x,r) = \{y : ||x - y|| = r\}$. The closure of the set $A \subset \mathbb{R}^n$ is denoted by $\mathrm{cl}(A)$. Put $\mathrm{cl}(B(x,r)) = \overline{B}(x,r)$. The n-dimensional Hausdorff measure is denoted by μ_n. The origin of $\mathbb{R}^n$ will be denoted by 0.

We shall use Lemma 1.8 on p. 10 of [F].

Lemma 1 *Let $\psi : E \to F$ be a surjective mapping such that $||\psi(x) - \psi(y)|| \leq c||x - y||$ $(x, y \in E)$ for a constant c. Then $\mu_s(F) \leq c^s \mu_s(E)$.*

The main result of this paper is the following theorem.

Theorem 1 *Assume that $\Omega \subset \mathbb{R}^n$ is open, and $f : \Omega \to \mathbb{R}$ is differentiable. Assume furthermore that $G \subset \mathbb{R}^n$ is open. Then $(\nabla f)^{-1}(G)$ is either empty or $\mu_1((\nabla f)^{-1}(G)) > 0$.*

We shall use Lemma 2 in the proof of our Theorem.

*Research supported by the Hungarian National Foundation for Scientific Research, Grant No. 2114.

Received by the editors March 17, 1992

Lemma 2 *Assume that $\Omega \subset \mathbb{R}^n$ is open, and $f : \Omega \to \mathbb{R}$ is differentiable. Assume furthermore that $x_2 \in \Omega$, $\delta > 0$, $B(x_2, \delta) \subset \Omega$, $\nu = ||\nabla f(x_2)|| < \eta$ and for any $y \in B(x_2, \delta)$ we have $||\nabla f(y)|| > 0$. Then $\mu_1((\nabla f)^{-1}(B(0, \eta))) > 0$.*

In the sequel first we prove the Theorem from Lemma 2. Then we provide a proof of Lemma 2.

(Proof of the theorem.) By subtracting a suitable linear function from f we can reduce the statement of the Theorem to the following one. *If $0 \in (\nabla f)^{-1}(B(0, \eta_1))$ for an $\eta_1 > 0$, then $\mu_1((\nabla f)^{-1} B(0, \eta_1)) > 0$.*

For a $\rho > 0$ put $H_\rho = \{x : ||\nabla f(x)|| < \rho\} = (\nabla f)^{-1}(B(0, \rho))$. For an $\eta \in (0, \eta_1)$ put $F_\eta = cl(H_\eta)$. Put $\eta_2 = \eta_1/2$.

Assume that F_{η_2} has an isolated point, say x_1. From $F_{\eta_2} = \text{cl}(H_{\eta_2})$ it follows that $x_1 \in H_{\eta_2}$, that is $\nu_1 = ||\nabla f(x_1)|| < \eta_2$. Since x_1 is an isolated point of F_{η_2} there exists a $\delta > 0$ such that $B(x_1, \delta) \subset \Omega$, and $\{x_1\} = B(x_1, \delta) \cap F_{\eta_2} = B(x_1, \delta) \cap H_{\eta_2}$. Then for any $y \in B(x_1, \delta) \setminus \{x_1\}$ we have $||\nabla f(y)|| \geq \eta_2 > 0$. If $\nu_1 \neq 0$ we can apply Lemma 2 with $\eta = \eta_2$, $x_2 = x_1$, $\nu = \nu_1$, $\delta = \delta$ and obtain $0 < \mu_1((\nabla f)^{-1}(B(0, \eta_2))) \leq \mu_1((\nabla f)^{-1}(B(0, \eta_1)))$ proving our Theorem.

If $\nu_1 = 0$ then choose a linear function g such that $||\nabla g(x)|| = \eta_2/4$. Put $f_1 = f + g$. Then $||\nabla f_1(x_1)|| = \eta_2/4$. For any $y \in B(x_1, \delta) \setminus \{x_1\}$ we have $||\nabla f_1(y)|| \geq \eta_2 - \frac{\eta_2}{4} > 0$. Thus Lemma 2 is applicable to f_1 with $\eta = \eta_2$, $x_2 = x_1$, $\nu = \eta_2/4$ and $\delta = \delta$. We obtain that $\mu_1((\nabla f_1)^{-1}(B(0, \eta_2))) > 0$. Since $f = f_1 - g$ we have $||\nabla f|| \leq ||\nabla f_1|| + ||\nabla g||$. Thus using that $\eta_1/2 = \eta_2$ we have $(\nabla f_1)^{-1}(B(0, \eta_2)) \subset (\nabla f)^{-1}(B(0, \eta_2 + \frac{\eta_2}{4})) \subset (\nabla f)^{-1}(B(0, \eta_1))$. This implies $\mu_1((\nabla f)^{-1}(B(0, \eta_1))) > 0$.

Assume that F_{η_2} does not have isolated points. Since the coordinate functions of ∇f are Baire-1 functions, there is a dense G_δ subset of F_{η_2}, say F', such that the restriction of ∇f onto F_{η_2} is continuous at the points of F'. Choose an $x_1 \in F'$. Assume that $||\nabla f(x_1)|| \neq 0$. Choose a $\delta > 0$ such that $B(x_1, \delta)$ is a subset of the domain of f and for any $y \in B(x_1, \delta) \cap F_{\eta_2}$ we have $||\nabla f(y)|| > 0$, this choice of δ is possible since $||\nabla f(x_1)|| \neq 0$ and the restriction of ∇f onto F_{η_2} is continuous at x_1. Since $H_{\eta_2} \subset F_{\eta_2}$ we have $||\nabla f(y)|| > 0$ for any $y \in B(x_1, \delta)$. Since $F_{\eta_2} = \text{cl}(H_{\eta_2})$ and x_1 is not an isolated point of F_{η_2} we can find an $x_2 \in H_{\eta_2} \cap B(x_1, \delta)$. Choose a δ_2 such that $B(x_2, \delta_2) \subset B(x_1, \delta)$. Then it is clear that the assumptions of Lemma 2 are satisfied with $\eta = \eta_2$, $x_2 = x_2$, $\nu = ||\nabla f(x_2)||$, $\delta = \delta_2$. Thus in this case our Theorem follows again from Lemma 2.

If $||\nabla f(x_1)|| = 0$ then, like in the corresponding case when x_1 was an isolated point, we can add to f a suitable linear function, g, which has a small gradient and obtain a function f_1. After this, the argument used for the case $||\nabla f(x_1)|| \neq 0$ is applicable to f_1. Finally an argument, similar to the one used for the case when x_1 was an isolated point, can show that $\mu_1((\nabla f)^{-1}(B(0, \eta_1))) > 0$. This concludes the proof of the Theorem.

(Proof of Lemma 2.) For $r \in [0, \delta)$ put $M(r) = \max\{f(x) : x \in C(x_2, r)\}$.

First we show that $M(r)$ is monotone increasing. Assume for a contradiction that one can find $0 < r_1 < r_2 < \delta$ such that $M(r_2) < M(r_1)$. Assume that f takes its absolute maximum on $\overline{B}(x_2, r_2)$ at y. Then $f(y) \geq M(r_1) > M(r_2)$. Thus y is in $B(x_2, r_2)$ and hence $||\nabla f(y)|| = 0$. This contradicts the assumption $||\nabla f(y)|| > 0$ for $y \in B(x_2, \delta)$.

As a monotone increasing function, $M(r)$ is almost everywhere differentiable and

$$\int_0^t M'(r)dr \leq M(t) - M(0) = M(t) - f(x_2)$$

holds for any $t \in (0, \delta)$ [cf. [S] Ch. IV., Th.7.4, p.119]. Since $||\nabla f(x_2)|| = \nu < \eta$ there exists a subset S of the interval $(0, \delta)$ such that $\mu_1(S) > 0$ and for $r \in S$ we have $M'(r) < \eta$. For any $r \in (0, \delta)$ choose an $x(r) \in C(x_2, r)$ such that $f(x(r)) = M(r)$. Observe that $x(r)$ is one-to-one and denote its inverse by ψ. By definition $M(r)$ is the maximum of f on $C(x_2, r)$ and this implies that $\nabla f(x(r))$ is perpendicular to $C(x_2, r)$. Assume for a contradiction that $\nabla f(x(r))$ points towards the interior of $\overline{B}(x_2, r)$. Using $||\nabla f(x(r))|| \neq 0$ the previous assumption implies that one can find a point $y \in B(x_2, r)$ such that $f(y) > f(x(r))$. From this it follows that $M(r') > M(r)$ for an $r' \in (0, r)$ chosen so that $y \in C(x_2, r')$. Since $M(r)$ is monotone increasing this is impossible. Therefore $\nabla f(x(r))$ points outwards of $\overline{B}(x_2, r)$. Denote by ℓ_1 the halfline starting at $x(r)$ pointing in the direction of $\nabla f(x(r))$. Furthermore denote by $y(t)$ the intersection point of ℓ_1 and $C(x_2, t)$. It is obvious that

$$\lim_{t \to r+} \frac{f(y(t)) - f(x(r))}{t - r} = ||\nabla f(x(r))||.$$

Since $f(x(r)) = M(r)$ and $f(y(t)) \leq M(t)$ we obtain that

$$||\nabla f(x(r))|| = \lim_{t \to r+} \frac{f(y(t)) - f(x(r))}{t - r}$$

$$\leq \lim_{t \to r+} \frac{M(t) - M(r)}{t - r} = M'(r) < \eta.$$

This is valid for any $r \in S$. It is easy to see that the mapping $\psi : x(r) \to r$ satisfies $||x(r_1) - x(r_2)|| \geq ||\psi(x(r_1)) - \psi(x(r_2))|| = |r_2 - r_1|$, and maps the set $\{x(r) : r \in S\}$ onto S, hence Lemma 1 implies that $\mu_1(\{x(r) : r \in S\}) \geq \mu_1(S) > 0$. Thus $\mu_1(\{x : ||\nabla f(x)|| < \eta\}) > 0$. This proves Lemma 2.

References

[F] K. J. Falconer *The geometry of fractal sets,* Cambridge University Press 1985.

[Q] Queries section, *Real Analysis Exchange*, Vol 16. No. 1., 1990-91, p. 373.

[R] C. A. Rogers, *Hausdorff Measures*, Cambridge University Press 1970.

[S] S. Saks, *Theory of the Integral,* Warsaw, 1937.