Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 214–220

M. Matejdes, Mathematics Department, Zvolen Technical University, Masarykova 24, 960 53 Zvolen, Czechoslovakia

Quasi-Continuous and Cliquish Selections of Multifunctions on Product Spaces

The notions of the quasi-continuity and cliquishness have been intensively studied for their close relation to continuity. The survey papers [2], [8], [11] contain among other things also results concerning separate and joint quasi-continuity as well as the fundamental results of the set of all continuity points of quasi-continuous functions and multifunctions. Studying these problems it is not possible to avoid other generalized continuity notions such as cliquishness, pointwise discontinuity and $\mathcal{B}$-continuity. From among the papers of this kind let us mention e.g. [1], [6], [7], [12]. Many results which are valid for functions may be transferred to multifunctions. The main aim of the present paper is to extend for multifunctions some known notions and techniques about cliquishness, pointwise discontinuity and quasi-continuity of functions defined on product spaces.

In what follows X, Y are topological spaces and M metric one. A multifunction $F : X \to \mathcal{K}(Y)$ is a set valued function which assigns to each element x of X a set $F(x) \in \mathcal{K}(Y) = \{A \subset Y : A \text{ is non-empty compact }\}$. A selection of F is any function $f : X \to Y$ such that $f(x) \in F(x)$ for any $x \in X$. If F is a multifunction defined on the product space $X \times Y$ we shall call an x-section for given $x \in X$ the multifunction $F_x(y) = F(x, y)$. The y-section F_y for a given $y \in Y$ is defined analogously

The upper (lower) inverse image $F^+(A)$ $(F^-(A))$ is defined for any set $A \subset Y$ as

$$F^+(A) = \{x \in X : F(x) \subset A\}, \quad F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$$

By $S_\varepsilon(A)$ we denote an ε-neighborhood of $A \subset M$, $\varepsilon > 0$ i.e., $S_\varepsilon(A) = \{z \in M : d(z, A) < \varepsilon\}$ where d is a metric for M. If $A = \{y\}$, we briefly write $S_\varepsilon(y)$. By $\operatorname{int}(B)$ we denote the interior of B.

Definition 1 *A multifunction $F : X \to \mathcal{K}(M)$ is said to be cliquish at a point $p \in X$ if for any $\varepsilon > 0$ and any neighborhood U of p there is a non-empty open*

Received by the editors February 20, 1992

set $B \subset U$ such that $\bigcap_{x \in B} S_\varepsilon(F(x)) \neq \emptyset$. F is cliquish if and only if it is cliquish at any point.

Remark 1 *(i) The condition $\bigcap_{x \in B} S_\varepsilon(F(x)) \neq \emptyset$ implies that there is a point $y \in M$ such that $S_\varepsilon(y) \cap F(x) \neq \emptyset$ for any $x \in B$.*

(ii) If a single valued function $f : X \to M$ is given, then under the natural interpretation of $f(x)$ as a one point set the above definition is equivalent to the usual definition of cliquishness of function [1].

(iii) The set of all points at which F is cliquish is closed. Consequently, F is cliquish if and only if it is cliquish on a dense set.

Definition 2 *Let d be a metric for M. A multifunction $F : X \to \mathcal{K}(M)$ is said to be h_d-cliquish at a point $p \in X$ (where h_d is the Hausdorff metric on $\mathcal{K}(M)$ induced by d) if for any $\varepsilon > 0$ and any neighborhood U of p there is a non-empty open set $B \subset U$ such that $h_d(F(a), F(b)) < \varepsilon$ for any $a, b \in B$.*

Let $\mathcal{B}$ be a family of subsets of X such that $\mathcal{G} \subset \mathcal{B} \subset \mathcal{B}r \cup \mathcal{G}$ where $\mathcal{G} = \{A \subset X : A$ is non-empty open $\}$ and $\mathcal{B}r = \{A \subset X : A$ is of second category having the Baire property $\}$.

Definition 3 *A multifunction $F : X \to \mathcal{K}(Y)$ is said to be u-$\mathcal{B}$-continuous (1-$\mathcal{B}$-continuous) at a point $p \in X$ if for any open sets V, U with $p \in U$, $F(p) \subset V$ $(F(p) \cap V \neq \emptyset)$ there is a set $B \in \mathcal{B}$ such that $B \subset U \cap F^+(V)$ $(B \subset U \cap F^-(V))$. F is u-$\mathcal{B}$-continuous (1-$\mathcal{B}$-continuous) if it is u-$\mathcal{B}$-continuous (1-$\mathcal{B}$-continuous) at any point [6]. For $\mathcal{B} = \mathcal{G}$ we have the well-known notion of upper (lower) quasi-continuity [9].*

F is said to be upper (lower) semi-continuous (briefly u.s.c. (l.s.c.)) at a point $p \in X$ if for any open set V such that $F(p) \subset V$ $(F(p) \cap V \neq \emptyset)$ we have $p \in int(F^+(V))$ $(p \in int(F^-(V)))$. F is said to be u.s.c. (l.s.c.) if it is u.s.c. (l.s.c.) at any point.

If a single valued function $f : X \to Y$ is given, then the notions of u-$\mathcal{B}$-continuity and 1-$\mathcal{B}$-continuity coincide and we simply refer to $\mathcal{B}$-continuity of f. Similarly, in respect of quasi-continuity and continuity of f.

Theorem 1 *Let X be Baire and M be separable metric. A multifunction $F : X \to \mathcal{K}(M)$ is cliquish if and only if it has a cliquish selection.*

Proof. It is clear that if f has a cliquish selection, then it is cliquish.

Now suppose F is cliquish. By [3, p.328, Th.3], there is a metric d for M, such that (M, d) is totally bounded. Let (M°, d°) be a completion of (M, d). By [3, p.337, Corollary of Th.19], M° is compact. Define $A(p) \subset M^\circ$ as follows:

$A(p) = \{z \in M^\circ$: for any open sets U, V with $p \in U$, $z \in V$ (V open in M°) there is a set $B \in \mathcal{G}$, $B \subset U$ such that $F(x) \cap V \neq \emptyset$ for any $x \in B\}$. We will show that $A(p)$ is non-empty. Let $\varepsilon(n) > 0$, $\varepsilon(n) \to 0$ and let $\mathcal{U}(p)$ be a complete system of neighborhoods of p. Since F is cliquish at p, for any $n = 1, 2, \ldots$ and any $U \in \mathcal{U}(p)$ there is a set $B(n, U) \in \mathcal{G}$, $B(n, U) \subset U$ and there is a point $y(n, U) \in M$ such that $F(x) \cap S_{\varepsilon(n)}(y(n, U)) \neq \emptyset$ for any $x \in B(n, U)$. Since M° is compact, there is a point $y \in M^\circ$ which is an accumulation point of net $\{y(n, U) : n = 1, 2, \ldots, U \in \mathcal{U}(p)\}$. It is clear that $y \in A(p)$. By [7, Th.5.4], F has a selection f which is quasi-continuous on a residual set. Thus f is cliquish.

Definition 4 *A multifunction $F : X \to \mathcal{K}(Y)$ is u-$\mathcal{D}$-continuous at a point $p \in X$ if for any open sets V, U with $F(p) \subset V$, $p \in U$ there is a set $A \subset U$ of second category such that $A \subset U \cap F^+(V)$. F is u-$\mathcal{D}$-continuous if it is u-$\mathcal{D}$-continuous at any point. Note that for any compact valued multifunction the set of its u-$\mathcal{D}$-discontinuity points is of first category [7, Remark 1.1] (Y is supposed to be second countable).*

Theorem 2 *Let $F : X \to \mathcal{K}(M)$ be a u-$\mathcal{D}$-continuous multifunction where X is Baire and M is separable metric. F has a quasi-continuous selection if and only if F is cliquish.*

Proof. It is clear that if F has a quasi-continuous selection, then F is cliquish. Now suppose that F is cliquish. By Theorem 1, there is a cliquish selection f of F. Now consider a multifunction $A : X \to M^\circ$ (M° as in the previous proof) defined as $A(p) = \{z \in M^\circ$: for any open sets U, V (V open in M°) with $z \in V$, $p \in U$ there is a non-empty open set H, $H \subset U$ such that $f(H) \subset V\}$. As in previous proof we can show that $A(p) \neq \emptyset$. Since $A(p)$ is closed, it is compact for any $p \in X$. We will show that A is u.s.c. Suppose A is not u.s.c. at a point p. That means there is an open set $V \supset A(p)$ (V open in M°) such that for any $U \in \mathcal{U}(p)$ there is a point $p(U) \in U$ such that $A(p(U)) \setminus V$ is non-empty. Let $y(U) \in A(p(U)) \setminus V$. Since M° is compact, there is a point $y \in M^\circ \setminus V$ which is an accumulation point of net $\{y(U) : U \in \mathcal{U}(p)\}$. Since $y \notin V$, $y \notin A(p)$. On the other hand it is easy to see that $y \in A(p)$ what is a contradiction. By [5, Th.1], the set S of continuity points of f is residual, hence $A(x) = \{f(x)\}$ for any $x \in S$.

Now we will show that any selection of A is quasi-continuous. Let g be a selection of A and let $p \in X$ and U, V be open (V open in M°) with $p \in U$, $g(p) \in V$. Since $g(p) \in A(p)$, there is a non-empty open set H such that $H \subset U$ and $f(H) \subset V$. $A(x) = \{g(x)\} = \{f(x)\}$ for any $x \in S$, hence $g(H \cap S) \subset V$. Thus g is $\mathcal{B}$-continuous. By [7, Th.2.5], g is quasi-continuous.

Now it is sufficient to show that $A(p) \cap F(p) \neq \emptyset$ for any $p \in X$. Suppose that $A(p) \cap F(p) = \emptyset$. Hence there are open disjoint sets G, W such that $G \supset A(p)$

and $W \supset F(p)$. Since A is u.s.c., $p \in \operatorname{int}(A^+(G))$. F is u-$\mathcal{D}$-continuous at p, hence there is a set T of second category such that $T \subset (\operatorname{int}(A^+(G))) \cap F^+(W)$. Thus for $x \in T \cap S$ we have $A(x) = \{f(x)\} \subset G$ and $F(x) \subset W$. Since $G \cap W = \emptyset$, we have a contradiction to the fact that f is selection of F.

One of the nicest result concerning the quasi-continuity is the Kempisty's theorem and its generalization. Under some general conditions on the spaces X, Y, M the quasi-continuity of x-sections and y-sections implies the quasi-continuity of $f : X \times Y \to M$. Another result in this direction was proved by Neubrunn [8] dealing with the quasi-continuity of multifunction. Roughly speaking the upper (lower) quasi-continuity of F_x and both upper and lower quasi-continuity of F_y implies the upper (lower) quasi-continuity of F. There are the examples in [8] that the lower (upper) quasi-continuity of F_y cannot be omitted. On the other hand as we shall prove in Theorem 4 the upper quasi-continuity of F_x and u-$\mathcal{B}$-continuity of F_y implies the existence of a quasi-continuous selection of F.

Another direction connected with the separate properties is the problem of finding the assumptions on spaces X, Y, M and the sections f_x, f_y such that $f : X \times Y \to M$ has at least one point of joint continuity. Recently published result [12, Th.2] is very close to that of [5, Th.3], assuming the cliquishness of f_x and the quasi-continuity of f_y.

A π-base (see [10, p.56], [12]) for a space $(Y, \mathcal{T})$ is a subset $\mathcal{P}$ of $\mathcal{T} \setminus \{\emptyset\}$ such that every non-empty set U of $\mathcal{T}$ contains a non-empty set G of $\mathcal{P}$.

Theorem 3 *Let X be a Baire space and Y be locally of π-countable type (i.e., each open non-empty subset of Y contains an open non-empty subset having a countable π-base). Let M be a metric space with a metric d and let $F : X \times Y \to \mathcal{K}(M)$. If F_x is h_d-cliquish for any $x \notin S$ ($S \subset X$ of first category) and F_y is u-$\mathcal{B}$-continuous for any $y \in Y$, then F is cliquish.*

Proof. Let $(p, q) \in U \times V \subset X \times Y$ where U and V are open neighborhood of p and q, respectively. Let $\varepsilon > 0$ be fixed. We can assume that V contains an open subset having a countable π-base $\{V(n)\}_{n=1}^{\infty}$. For any $n = 1, 2, \ldots$ denote $T(n) = \{x \in U \cap (X \setminus S)$: there is an open set $V(x)$ with $V(n) \subset V(x) \subset V$ and for any $r, w \in V(x)$ we have $h_d(F_x(r), F_x(w)) < \varepsilon/10\}$. Since $\bigcup_{k=1}^{\infty} T(n) = U \cap (X \setminus S)$, there is an index N and a non-empty open set $J \subset U$ such that $J \cap T(N)$ is of second category at every point of J. Let $(a, b) \in (J \cap T(N)) \times V(N)$. We can suppose that F_b is l.s.c. at a. It is possible, because F_b is l.s.c. on a residual set, by [6, Th.1]. Since F_b is u-$\mathcal{B}$-continuous and l.s.c. at a, there is a set $B = (H \setminus I) \cup E \in \mathcal{B}$, such that $B \subset J, H$ is open, I, E are of first category and

$$h_d(F_b(x),\ F_b(a)) < \varepsilon/10 \text{ for any } x \in B. \tag{1}$$

Let $T = (H \setminus I) \cap J \cap T(N)$. T is of second category at any point of $H \cap J$. Further $T \times V(N)$ is dense in $(H \cap J) \times V(N) \subset U \times V$. Let $(s,t) \in T \times V(N)$, $(u,v) \in (H \cap J) \times V(N)$ be arbitrary. Since F_v is u-$\mathcal{B}$-continuous at u, there is a set $D \subset H \cap J$, $D \in \mathcal{B}$ of second category with the Baire property such that $F_v(x) \subset S_{\varepsilon/10}(F_v(u))$ for any $x \in D$. Choose $z \in D \cap T$. Thus

$$F_v(z) \subset S_{\varepsilon/10}(F_v(u)). \tag{2}$$

We have the following inequalities:

$$\text{Since } z \in T(N), \quad h_d(F_z(v),\ F_z(b)) < \varepsilon/10.$$

$$\text{By (1)}, \quad h_d(F_b(z),\ F_b(a)) < \varepsilon/10$$

$$h_d(F_b(a),\ F_b(s)) < \varepsilon/10.$$

$$\text{Since } s \in T(N), \quad h_d(F_s(b),\ F_s(t)) < \varepsilon/10.$$

$$\text{Thus} \quad h_d(F_z(v),\ F_s(t)) < (4/10)\varepsilon = (2/5)\varepsilon.$$

By (2), $F_v(z) \subset S_{\varepsilon/10}(F_v(u))$, hence $F_s(t) \subset S_\varepsilon(F_v(u))$.
We have proved that for any $(p,q) \in X \times Y$, for any neighborhood $U \times V$ of (p,q) and any $\varepsilon > 0$ there is an open set $G \subset U \times V$ $(G = (H \cap J) \times V(N))$ and there is a set A $(= F(s,t))$ such that $A \subset S_\varepsilon(F(u,v))$ for any $(u,v) \in G$. Hence $\emptyset \neq A \subset \bigcap_{(u,v)\in G} S_\varepsilon(F(u,v))$. Thus F is cliquish.

By [7, Th.2.5], a function $f : X \to M$ (X – Baire) is quasi-continuous iff it is $\mathcal{B}$-continuous. Thus we have the following consequence of Theorem 3 which is a generalization of [5, Th.3] as well as [12, Th.2].

Corollary 1 *Using the same conditions on X, Y and M as in Theorem 3, the cliquishness of x-sections (except for a set of first category) and the quasi-continuity of y-sections implies the cliquishness of a function $f : X \times Y \to M$. Moreover, if $X \times Y$ is Baire, f is continuous on a residual set, by [5, Th.1].*

Corollary 2 *Let X be Baire and Y be locally of π-countable type such that $X \times Y$ is Baire. Further let M be a separable metric space and let $F : X \times Y \to \mathcal{K}(M)$ be a multifunction such that F_x is h_d-cliquish for any x (except possibly for a set of first category) and F_y is u-$\mathcal{B}$-continuous for any $y \in Y$. Then F has a cliquish selection.*

Proof. It follows directly from Theorems 1 and 3.

Theorem 4 *Let X be a Baire space and Y be Baire locally π-countable type. Let M be a separable metric space. If $F : X \times Y \to \mathcal{K}(M)$ is a multifunction such that F_x is upper quasi-continuous for any $x \notin S$ (S is of first category) and F_y is u-$\mathcal{B}$-continuous for any $y \in Y$, then F has a quasi-continuous selection.*

Proof. By [6, Th.1] and [9, Th.3.1.7], F_x ($x \notin S$) is l.s.c. and u.s.c. except for a set of first category, hence F_x is h_d-cliquish.

It is sufficient to show that F is u-$\mathcal{D}$-continuous. Let $(x,y) \in X \times Y$ and let $U \times V$, G be open with $(x,y) \in U \times V$, $F(x,y) \subset G$. Since F_y is u-$\mathcal{B}$-continuous, there is a set $T \subset U$ of second category such that $F_y(t) \subset G$ for any $t \in T$. Further for any $t \in T \setminus S$, F_t is upper quasi-continuous, hence there is an open set $C(t) \subset V$ such that $\{t\} \times C(t) \subset F^+(G)$. Then an application of the Kuratowski-Ulam theorem (which can be proved for X, Y by similar way as in [10, p.56]) guarantees that $F^+(G) \cap (U \times V)$ is of second category.

By [10, Th. 15.3], $X \times Y$ is Baire and the proof follows from Theorems 2 and 3.

Example 1 *Let $X = \langle 0,1 \rangle$ with the usual topology and A, B be countable dense disjoint subsets of X. Define $G : X \to X$ as $G(x) = \{0\}$ if $x \in A$, $G(x) = \{1\}$ if $x \in B$ and $G(x) = \langle 0,1 \rangle$ for $x \notin A \cup B$. Let $\mathcal{B} = \{A \subset X : A$ is of second category having the Baire property$\}$. G is 1-$\mathcal{B}$-continuous, but it is not h_d-cliquish. Considering $F : X \times \{0\} \to X$ defined as $F([x,0]) = G(x)$ we can easy to see that in Theorem 3 it is not possible to replace u-$\mathcal{B}$-continuity of F_y by 1-$\mathcal{B}$-continuity. Similar situation arises in Theorem 4.*

Consider X, Y, M and F from Theorem 4.

Problem 1 *Does F have the lower (upper) Baire property? (that means does $F^-(G)(F^+(G))$ have the Baire property for any open set G?).*

Let $\mathcal{L}$ be a set of all quasi-continuous selections of F. Define $G : X \times Y \to \mathcal{K}(M)$ as $G(p) = cl(\cup_{f \in \mathcal{L}}\{f(p)\})$ where cl denotes the closure. It is clear that $G(p) \subset F(p)$ for any $p \in X \times Y$ and G is lower quasi-continuous. What is the Baire category of $A = \{p \in X \times Y : G(p) = F(p)\}$?

Problem 2 *In [4] cliquishness of multifunctions taking values in uniform spaces have been studied. Can key results of our paper be appropriately generalized to the cliquishness in the sense of the paper [4]?*

References

[1] W.W. Bledsoe, Neighbourly functions, PAMS, 3 (1952), 114-115.

[2] V. Baláž, On weak forms of continuity and product spaces, Acta Math. Univ. Con., 44-45 (1984), 169-179.

[3] R. Engelking, Topologia ogólna, PWN, Warszawa, 1976.

[4] J. Ewert, On quasi-continuous and cliquish maps with values in uniform spaces, Bull. Ac. Pol. Math. 32 (1984), 81-88.

[5] L.A. Fudali, On cliquish functions on product spaces, Math. Slovaca, 33 (1983), 53-58.

[6] M. Matejdes, Quelques remarques sur la quasi-continuité des multifonctions, Math. Slovaca, 37 (1987), 267-271.

[7] M. Matejdes, Sur les selécteurs des multifonctions, Math. Slovaca, 37 (1987), 111-124.

[8] T. Neubrunn, On quasi-continuity of multifunctions, Math. Slovaca, 32 (1982), 147-154.

[9] T. Neubrunn, Quasi-continuity, Real Anal. Exchange, 14 (1988-89), 259-306.

[10] J.C. Oxtoby, Measure and category, Springer, 1971.

[11] Z. Piotrowski, Separate and joint continuity, Real Anal. Exchange, 11 (1985-86), 293-322.

[12] Z. Piotrowski and E. Wingler, A note on continuity points of functions, Real Anal. Exchange, 16 (1990-91), 408-414.