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Topologies Between Compact and Uniform Convergence on Function Spaces, II

1. Introduction

The strict topology on the set $C^*(X)$ of all bounded continuous real-valued functions on a space X was introduced by Buck [1] for the case that X is a locally compact Hausdorff space. Other authors have extended and studied this topology on $C^*(X)$ for X an arbitrary Tychonoff space (see, for example, [3], [4], [8], [9] and [10]). In [4], Gulick introduced the σ-compact-open topology on $C^*(X)$, where convergence is uniform on all σ-compact subsets of X. This latter topology was independently studied in [5] where the definition was further extended to the set $C(X)$ of all continuous real-valued functions on a Tychonoff space X. In the presence of local compactness, Buck observed in [1] that the compact-open topology on $C^*(X)$ can be generated by a collection of seminorms induced by the collection of continuous functions with compact support. The purpose of this present work is twofold. The first one is to extend the strict topology on $C^*(X)$ to $C(X)$ and study it. The second one is to describe the topology of uniform convergence on σ-compact subsets in terms of seminorms arising from a nice subclass of bounded continuous functions. To do this, we introduce a new topology τ (to be defined in the next section) on $C(X)$ and study it extensively. The studies of the strict and the τ topologies are closely related.

Throughout the rest of the paper, we use the following conventions. All spaces are Tychonoff spaces, and whenever we deal with local compactness, we

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mean a locally compact Hausdorff space. If X and Y are any two spaces with the same underlying set, then we use $X = Y$, $X \leq Y$ and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X. The symbols $\mathbb{R}$ and $\mathbb{N}$denote the spaces of real numbers and natural numbers, respectively. Finally, the constant zero-function in $C(X)$ is denoted by f_0.

2. Basic definitions and notations

Let $B(X)$ be the set of all bounded real-valued functions on X. A function f in $B(X)$ is said to ***vanish at infinity*** if for every $\varepsilon > 0$ the subset $\{x \in X \,|\, |f(x)| \geq \varepsilon\}$ is compact. Define $B_0(X) = \{f \in B(X)\ f \text{ vanishes at infinity}\}$ and $C_0(X) = B_0(X) \cap C(X)$. Clearly $C^*(X) = B(X) \cap C(X)$.

A subset A of X is called ***almost σ-compact*** if there exists a σ-compact subset B of A such that B is dense in A. An element $f \in B(X)$ has a ***compact*** (respectively, an ***almost σ-compact***) ***support*** if there exists a compact (respectively, an almost σ-compact) subset K of X such that $f = 0$ on $X - K$. Let $B_{00}(X) = \{\phi \in B(X)\ \phi \text{ has a compact support}\}$, $B_1(X) = \{\phi \in B(X)\ \phi \text{ has an almost } \sigma\text{-compact support}\}$, $C_{00}(X) = B_{00}(X) \cap C(X)$, and $C_1(X) = B_1(X) \cap C(X)$.

By a ***pseudo-seminorm*** on a real linear space E is meant a real-valued function p on E such that

(1) $p(0) = 0$

(2) $p(x) = p(-x)$ for all $x \in E$, and

(3) $p(x + y) \leq p(x) + p(y)$ for all x, $y \in E$.

A pseudo-seminorm p is called a ***seminorm*** if

(4) $p(tx) = |t|p(x)$ for all $x \in E$ and $t \in \mathbb{R}$

is also true.

To each $\phi \in B(X)$ define the pseudo-seminorm p_ϕ on $C(X)$ by

$$p_\phi(f) = \min\{1, \sup\{|\phi(x)f(x)|\ x \in X\}\}.$$

Also for each nonempty subset A of X, define the pseudo-seminorm p_A on $C(X)$ by

$$p_A(f) = \min\{1, \sup\{|f(x)|\ x \in A\}\}.$$

On $C^*(X)$, however, we can use the seminorms p_ϕ and p_A defined by

$$p_\phi(f) = \sup\{|\phi(x)f(x)|\ x \in X\} \text{ and } p_A(f) = \sup\{|f(x)|\ x \in A\},$$

respectively.

Let $C_k(X)$ denote the space $C(X)$ equipped with the ***compact-open topology*** k which is generated by the collection of seminorms $\{p_\phi \ \phi \in B_{00}(X)\}$.

The ***strict topology*** (or ***β-topology***) on $C(X)$ is generated by the collection of pseudo-seminorms $\{p_\phi \ \phi \in B_0(X)\}$ and is denoted by $C_\beta(X)$. Sentilles [9] and Summers [10] called this topology on $C^*(X)$ the substrict topology and denoted it by β_0, Gulick [4] called it the strict topology and used the notation t_s. The notation β is due to Buck [1].

The ***uniform topology*** u on $C(X)$ is generated by the complete metric ρ, where for $f,\ g \in C(X)$,

$$\rho(f,g) = p_X(f-g) = \min\{1, \sup\{|f(x) - g(x)| \ x \in X\}\},$$

and the corresponding topological space is denoted by $C_u(X)$. It is easy to see that u can also be generated by the collection of pseudo-seminorms $\{p_\phi \ \phi \in \mathcal{F}\}$ where $\mathcal{F}$ is either $B(X)$ or $C^*(X)$. The topology u on $C^*(X)$ is actually generated by the complete supremum norm.

Let $C_{\sigma,u}(X)$ be the space equipped with the ***topology of uniform convergence on σ-compact subsets of*** X. This topology, denoted by (σ, u), is generated by the collection of pseudo-seminorms $\{p_A \ A$ is a σ-compact subset of $X\}$, and is studied in [5]. For each $A \subset X$ and $\varepsilon > 0$, let $V_{A,\varepsilon} = \{f \in C(X) \ p_A(f) < \varepsilon\}$. Then the collection

$$\{V_{A,\varepsilon} \ A \text{ is a } \sigma\text{-compact subset of } X \text{ and } \varepsilon > 0\}$$

forms a neighborhood base at f_0 for (σ, u). The subspace $C^*_{\sigma,u}(X)$ was introduced in [4], where it was shown that $C^*_\beta(X) \leq C^*_{\sigma,u}(X)$. The following two facts can be found in the papers [4] and [5].

(1) $C^*_{\sigma,u}(X) = C^*_u(X)$ if and only if X contains a dense σ-compact subset. This is also true if $C^*(X)$ is replaced by $C(X)$.

(2) $C_k(X) = C_{\sigma,u}(X)$ if and only if every σ-compact subset of X has compact closure.

It may be mentioned here that in [5] a set-open topology on $C(X)$ is introduced called the ***σ-compact-open topology***, and this space is denoted by $C_\sigma(X)$. In general, $C_\sigma(X) \leq C_{\sigma,u}(X)$, and $C_\sigma(X) = C_{\sigma,u}(X)$ if and only if X is pseudocompact. Since $C^*_\sigma(X) = C^*_{\sigma,u}(X)$ is always true, the topology of uniform convergence on σ-compact subsets $C^*(X)$ may be called the σ-compact-open topology, as was done in [4].

It is not difficult to show that the topology (σ, u) on $C(X)$ is generated by the collection of pseudo-seminorms $\{p_\phi \ \phi \in B_1(X)\}$. For the justification, see

Theorem 3.3 of the next section. We would like to describe it in terms of pseudo-seminorms arising from a subclass of bounded continuous functions. While probing this possibility, a new topology τ on $C(X)$ naturally crops up. The space $C(X)$ equipped with the topology generated by the collection of pseudo-seminorms $\{p_\phi \, \phi \in C_1(X)\}$ is denoted by $C_\tau(X)$. For each $\phi \in B(X)$ and $\varepsilon > 0$, let $V_{\phi,\varepsilon} = \{f \in C(X) \, p_\phi(f) < \varepsilon\}$. Then the collection $\{V_{\phi,\varepsilon} \, \varepsilon \in \mathcal{D}, \ \varepsilon > 0\}$ forms a neighborhood base at f_0 for β or τ according as $\mathcal{D} = B_0(X)$ or $\mathcal{D} = C_1(X)$, respectively.

We call a space *locally almost σ-compact* if it is a Tychonoff space such that each point has an almost σ-compact neighborhood. It can be shown that $C_k(X) \leq C_\tau(X)$ if and only if X is locally almost σ-compact. The property of local almost σ-compactness ensures the non-triviality of $C_1(X)$, so that whenever needed, we assume X to be locally almost σ-compact while working with τ. Let us observe that if X is almost σ-compact, then $C_1(X) = C^*(X)$, which implies that $C_\tau(X) = C_u(X)$. Thus the proper setting for the study of $C_\tau(X)$ should be on a space X which is not almost σ-compact but at the same time is locally almost σ-compact. Note that a locally compact space is locally almost σ-compact. In the next section we give an example of a space which is locally almost σ-compact, but is neither almost σ-compact nor locally compact.

3. Comparison of topologies

From the definitions it is evident that $C_k(X) \leq C_\beta(X) \leq C_u(X)$. The comparison of the neighborhood systems at f_0 shows that $C_\beta(X) \leq C_{\sigma,u}(X)$ and $C_\tau(X) \leq C_{\sigma,u}(X)$. It has been observed by Giles [3], Gulick [4] and others that in the presence of local compactness, the strict topology β on $C^*(X)$ can also be generated by the collection of seminorms $\{p_\phi \, \phi \in C_0(X)\}$. Since for a locally compact space $C_0(X) \subset C_1(X)$, from the previous observation it follows that for such a space $C_\beta(X) \leq C_\tau(X)$. In this section, we determine when these inequalities are equalities and give examples to illustrate the differences. The proofs of (a), (b) and (c) of the next theorem can be found in [4].

Theorem 3.1 *The following are true.*

(a) $C_k(X) = C_\beta(X)$ if and only if the closure of each σ-compact subset of X is compact.

(b) $C_\beta(X) = C_u(X)$ if and only if X is compact.

(c) $C_\beta(X) = C_{\sigma,u}(X)$ if and only if the closure of each σ-compact subset of X is compact.

(d) If X is locally compact, then $C_\beta(X) = C_\tau(X)$ if and only if $C_0(X) = C_1(X)$.

Since $C_1(X) = C^*(X)$ if and only if X is almost σ-compact, it follows that $C_\tau(X) = C_u(X)$ if and only if X is almost σ-compact. The next two results determine the conditions for the equality of τ and (σ, u).

Lemma 3.2 *Let A be a nonempty σ-compact subset of X, let $\phi \in C_1(X)$ and let $0 < \varepsilon, \delta \leq 1$. Then $\{f \in C(X)\, p_\phi(f) < \varepsilon\} \subset \{f \in C(X)\, p_A(f) < \delta\}$ if and only if $\inf\{|\phi(x)|\, x \in A\} \geq \frac{\varepsilon}{\delta}$.*

Proof. First suppose $\inf\{|\phi(x)|\, x \in A\} \geq \frac{\varepsilon}{\delta}$ and $g \in \{f \in C(X)\, p_\phi(f) < \varepsilon\}$. Then define $\varepsilon' = \sup\{|\phi(x)g(x)|\, x \in X\}$. Since $\varepsilon' < \varepsilon$, there exists some $\delta' < \delta$ such that $\varepsilon' \leq \delta' \inf\{|\phi(x)|\, x \in A\}$. For each $a \in A$, since $\phi(a) \neq 0$, we have

$$|g(a)| \leq \frac{\delta' \inf\{|\phi(x)|\, x \in A\}}{|\phi(a)|} \leq \delta'.$$

Therefore $p_A(g) \leq \delta' < \delta$.

Conversely, suppose $\{f \in C(X)\, p_\phi(f) < \varepsilon\} \subset \{f \in C(X)\, p_A(f) < \delta\}$. We first show that $\phi(a) \neq 0$ for all $a \in A$. If not, there is some $x_0 \in A - \phi^{-1}(\mathbb{R} - \{0\})$. Then there exists an open neighborhood U of x_0 in X such that $|\phi(x)| < \frac{\varepsilon}{2\delta}$ for all $x \in U$. Now choose continuous $g\, X \to [0, \delta]$ such that $g(x_0) = \delta$ and $g(x) = 0$ for all $x \in X - U$. Since $g(x_0) = \delta$, we have $g \notin \{f \in C(X)\, p_A(f) < \delta\}$. If $x \in U$, then $|\phi(x)g(x)| < \frac{\varepsilon}{2}$; while if $x \in X - U$, then $|\phi(x)g(x)| = 0$. Therefore $g \in \{f \in C(X)\, p_\phi(f) < \varepsilon\}$, and we have a contradiction.

Next suppose, again by way of contradiction, that $\inf\{|\phi(x)|\, x \in A\} < \frac{\varepsilon}{\delta}$. Then there exists some $a \in A$ such that $\delta|\phi(a)| < \varepsilon$. Since $\phi(a) \neq 0$, we may define $g \in C^*(X)$ by

$$g(x) = \begin{cases} \frac{\delta|\phi(a)|}{|\phi(x)|}, & \text{if } |\phi(x)| \geq |\phi(a)|, \\ \delta, & \text{if } |\phi(x)| \geq |\phi(a)|. \end{cases}$$

Now $g(a) = \delta$, so that $g \notin \{f \in C(X)\, p_A(f) < \delta\}$. On the other hand, for each $x \in X$, $|\phi(x)g(x)| \leq \delta|\phi(a)|$. Hence $g \in \{f \in C(X)\, p_\phi(f) < \varepsilon\}$, which is a contradiction. ∎

Theorem 3.3 *For every space X, $C_\tau(X) = C_{\sigma,u}(X)$ if and only if for every nonempty σ-compact subset A of X there exists a σ-compact subset B of X and a continuous function $\phi\, X \to [0,1]$ such that $\phi = 1$ on A and $\phi = 0$ on $X - \overline{B}$.*

Proof. First suppose $C_\tau(X) = C_{\sigma,u}(X)$, and let A be a σ-compact subset of X. We then have $\{f \in C(X)\, p_\psi(f) < \varepsilon\} \subset \{f \in C(X)\, p_A(f) < 1\}$ for some

$\psi \in C_1(X)$ and $\varepsilon > 0$. So there is a σ-compact subset B of X such that $\psi(X - \overline{B}) = 0$. By 3.2, we have $\inf\{|\psi(x)| \, x \in A\} \geq \varepsilon$. Now define continuous $\phi \, X \to [0, 1]$ by $\phi(x) = \min\left\{1, \frac{|\psi(x)|}{\varepsilon}\right\}$. Clearly $\phi(A) = 1$ and $\phi(X - \overline{B}) = 0$, as desired.

For the converse, suppose $\{f \in C(X) \, p_A(f) < \delta\}$ is a given basic neighborhood of f_0 in $C_{\sigma,u}(X)$, where $0 < \delta \leq 1$. By hypothesis, there exists a σ-compact subset B of X and a continuous $\phi \, X \to [0, 1]$ such that $\phi(A) = 1$ and $\phi(X - \overline{B}) = 0$. Since $\phi \in C_1(X)$ and $\inf\{|\phi(x)| \, x \in A\} = 1$, by 3.2, we have $\{f \in C(X) \, p_\phi(f) < \delta\} \subset \{f \in C(X) \, p_A(f) < \delta\}$. Therefore $C_\tau(X) = C_{\sigma,u}(X)$. ∎

For the next two results, we need the following definition. A set S is said to be ***regularly σ-compact*** if $S = \bigcup_{n=1}^{\infty} K_n$ where for each n, K_n is compact and $K_n \subset \operatorname{Int} K_{n+1}$. In a locally compact space, every σ-compact subset is contained in a regularly σ-compact subset. Every regularly σ-compact set is an open F_σ-set.

Corollary 3.4 *If X is locally compact and paracompact, then $C_\tau(X) = C_{\sigma,u}(X)$.*

Proof. If X is locally compact and paracompact, then the closure of every σ-compact subset is σ-compact (see [2], page 382). Let A be a σ-compact subset of X. Then $\overline{A}$ is also σ-compact, and there exists a regularly σ-compact subset B containing $\overline{A}$. By the normality of X, we have a continuous function $\phi \, X \to [0, 1]$ such that $\phi = 1$ on A and $\phi = 0$ on $X - \overline{B}$. ∎

In the absence of local compactness, $C_\beta(X)$ and $C_\tau(X)$ cannot be compared. But for a locally compact normal space, we have the following necessary condition for these spaces to be equal.

Theorem 3.5 *Let X be a locally compact normal space. If $C_\beta(X) = C_\tau(X)$, then every closed σ-compact subset of X is compact.*

Proof. The hypothesis implies that $C_0(X) = C_1(X)$. Now let A be a closed σ-compact subset of X, so that there exists a regularly σ-compact subset S containing A. By the normality of X, there exists a continuous function $f \, X \to [0, 1]$ such that $f(A) = 1$ and $f(X - S) = 0$. For $0 < \varepsilon < 1$, there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \in X - K$. Hence $A \subset K$, and consequently A is compact. ∎

We end this section by looking at some examples.

Example 3.6 *If X is any compact space, then*

$$C_k(X) = C_\beta(X) = C_\tau(X) = C_{\sigma,u}(X) = C_u(X).$$

Example 3.7 *If X is any uncountable discrete space, then*

$$C_k(X) < C_\beta(X) < C_\tau(X) = C_{\sigma,u}(X) < C_u(X).$$

Example 3.8 *If $X = \mathbb{N}$ or $X = \mathbb{R}$, then*

$$C_k(X) < C_\beta(X) < C_\tau(X) = C_{\sigma,u}(X) = C_u(X).$$

Example 3.9 *If X is the space $[0,\omega_1)$ of countable ordinals with the order topology, then*

$$C_k(X) = C_\beta(X) = C_\tau(X) = C_{\sigma,u}(X) < C_u(X).$$

Example 3.10 *Let X be the deleted Tychonoff plank, that is, $X = [0,\omega_1] \times [0,\omega_0] - \{(\omega_1,\omega_0)\}$. Then X is not normal but is locally compact and almost σ-compact. The set $\{\omega_1\} \times [0,\omega_0)$ is a closed σ-compact subset of X which is not compact. Therefore*

$$C_k(X) < C_\beta(X) < C_\tau(X) = C_{\sigma,u}(X) = C_u(X).$$

Example 3.11 *Let X be the topological sum of the space of rationals with the usual topology and the space of irrationals with the discrete topology. Then X is neither locally compact nor almost σ-compact, but it is locally almost σ-compact. Note that the closure of every σ-compact subset is σ-compact. It is easy to see that given a nonempty σ-compact subset A, there exist a σ-compact subset B of X and a continuous function $\phi\, X \to [0,1]$ such that $\phi = 1$ on A and $\phi = 0$ on $X - \overline{B}$. It now follows that*

$$C_k(X) < C_\beta(X) \text{ and } C_k(X) < C_\tau(X) = C_{\sigma,u}(X) < C_u(X).$$

Example 3.12 *Let X be an uncountable space in which all points are isolated except for a distinguished point s, where a neighborhood of s is any set containing s whose complement is countable. All compact subsets of X are finite, so that the compact-open topology on $C(X)$ coincides with the topology of pointwise convergence. The σ-compact subsets of X are countable and closed, and in fact s does not have an almost σ-compact neighborhood; so that X is not locally almost σ-compact. This means that k and τ on $C(X)$ are not comparable. It is easy to see that $C_1(X) = \{f \in C(X)\, f(s) = 0\}$ and $C_1(X)$ is non-trivial. Also using Theorem 3.3, we see that*

$$C_\tau(X) < C_{\sigma,u}(X) < C_u(X).$$

Example 3.13 *Let $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$ where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. Define $X = \mathbb{N}^* - \{p\}$ where p is a point of $\mathbb{N}^*$ which is not a P-point (see [7]). Then X is a locally compact space which is not paracompact and does* *not have a dense σ-compact subset (since points are not G_δ-sets). Also since p is not a P-point, there is some σ-compact subset of X which has non-compact closure. Therefore*

$$C_k(X) < C_\beta(X) \leq C_\tau(X) \leq C_{\sigma,u}(X) < C_u(X).$$

Note that $C_\beta(X) < C_{\sigma,u}(X)$, so that at best, only one of the two equalities $C_\beta(X) = C_\tau(X)$ and $C_\tau(X) = C_{\sigma,u}(X)$ can hold. We do not know of an example of a locally almost σ-compact space for which τ is strictly weaker than (σ, u), however this example might have that property.

4. Linear topological structure

It is easy to show that $C_\beta(X)$, $C_\tau(X)$ and $C_{\sigma,u}(X)$ are topological groups under usual addition. Hence they are uniformizable and consequently completely regular. Since β and (σ, u) are finer than k, they are Hausdorff and consequently $C_\beta(X)$ and $C_{\sigma,u}(X)$ are Tychonoff spaces. If X is a locally almost σ-compact space, then $C_\tau(X)$ is also a Hausdorff space. But $C_\tau(X)$ may be a Hausdorff space without X being locally almost σ-compact. Example 3.12 provides an example of such a space. We now determine when $C_\beta(X)$, $C_\tau(X)$ and $C_{\sigma,u}(X)$ are linear topological spaces; that is, when scalar multiplication is continuous on these spaces.

Theorem 4.1 *For every space X, $C_\beta(X)$ is a linear topological space if and only if X is pseudocompact.*

Proof. If X is pseudocompact, then $C_\beta(X) = C^*_\beta(X)$, and the latter is a locally convex space. Conversely, suppose X is not pseudocompact. Then X contains a closed C-embedded copy A of $\mathbb{N}$; say $A = \{x_n\ n \in \mathbb{N}\}$. Define $\phi\ X \to \mathbb{R}$ and $f\ A \to \mathbb{R}$ by $\phi(x) = \frac{1}{n}$ for $x = x_n$, $\phi(x) = 0$ for $x \in X - A$, and $f(x_n) = n^2$ for $n \in \mathbb{N}$. Note that $\phi \in B_0(X)$. If $F \in C(X)$ is any continuous extension of f, then for all $n \in \mathbb{N}$, $|\phi(x_n)\frac{1}{n}F(x_n)| = 1$; so that $\frac{1}{n}F \notin \{g \in C(X)\ p_\phi(g) < 1\}$ for any $n \in \mathbb{N}$. This shows that scalar multiplication is not continuous at $(0, F)$. ∎

If X is pseudocompact, then clearly $C_\tau(X) = C^*_\tau(X)$, which is a locally convex space. For the converse, we have the following result.

Theorem 4.2. *Let X be a locally almost σ-compact normal space. If $C_\tau(X)$ is a linear topological space, then X is pseudocompact.*

Proof. If X is not pseudocompact, then choose A as in the previous theorem. Since X is locally almost σ-compact, there exists an almost σ-compact subset B of X such that $A \subset \operatorname{Int} B$. By the normality of X, there exists a continuous function $\phi\ X \to [0,1]$ such that $\phi = 1$ on A and $\phi = 0$ on $X - \mathrm{Int}\,B$. Note that $\phi \in C_1(X)$. Define $f\ A \to \mathbb{R}$ by $f(x_n) = n$ for $n \in \mathbb{N}$, and extend f to a continuous F on X. Now the arguments similar to those in the previous theorem show that scalar multiplication is not continuous at $(0, F)$. ∎

It follows as consequences of Theorem 1.1 in [5] that $C_{\sigma,u}(X)$ and $C_u(X)$ are linear topological spaces if and only if X is pseudocompact.

5. Completeness

If E is a topological group, a net $\{x_\alpha\}$ in E is called a *Cauchy net* provided that for each neighborhood U of 0 in E there exists α_0 such that $x_\alpha - x_{\alpha'} \in U$ for all α, $\alpha' \geq \alpha_0$. Then E is *complete* if every Cauchy net in E converges to some element in E.

A Tychonoff space X is called a *k_R-space* (respectively, *k_f-space*) provided that every real-valued (respectively, bounded real-valued) function which is continuous on each compact subspace is necessarily continuous on all of X. Obviously, a k-space is a k_R-space and a k_R-space is a k_f-space.

Theorem 5.1 *For every space X, $C^*_\beta(X)$ is closed in $C_\beta(X)$.*

Proof. To show that $C(X) - C^*(X)$ is open, let $f \in C(X) - C^*(X)$. Then f is unbounded, so that we can obtain a sequence $\{x_n\ n \in \mathbb{N}\}$ in X such that $f(x_n) \geq 2^{n+1}$ for each n. Define $\phi \in B_0(X)$ by $\phi(x) = \frac{1}{2^n}$ if $x = x_n$ for some $n \in \mathbb{N}$ and $\phi(x) = 0$ otherwise. Choose $g \in C(X)$ so that $p_\phi(f - g) < 1$. Then for each n, $|(f(x_n) - g(x_n))\phi(x_n)| < 1$. This implies that $g(x_n) > 2^n$, so that $g \in C(X) - C^*(X)$. Therefore $C(X) - C^*(X)$ is indeed a neighborhood of f in $C_\beta(X)$. ∎

Since $C^*_\beta(X)$ is complete if and only if X is a k_f-space (see [9]), we have the following corollary.

Corollary 5.2 *If $C_\beta(X)$ is complete, then X is a k_f-space.*

In order to examine the converse of Corollary 5.2, we consider the topology of uniform convergence extended to the set $F(X)$ of all real-valued functions on X. Let

$$V(f,\varepsilon) = \{g \in F(X)\ |f(x) - g(x)| < \varepsilon \text{ for all } x \in X\}$$

where $f \in F(X)$ and $\varepsilon > 0$. To generate the topology of uniform convergence on $F(X)$, take $\{V(f,\varepsilon)\ \varepsilon > 0\}$ as a neighborhood base at f; and again denote this topology by u as in the case of $C(X)$. It is easy to show that the topology u on each of $F(X)$, $C(X)$ and $C^*(X)$ is complete.

Theorem 5.3 *If X is a k_R-space, then $C_\beta(X)$ is complete.*

Proof. Let $\{f_\alpha\}$ be a Cauchy net in $C_\beta(X)$. Then $\{f_\alpha\}$ is also a Cauchy net in $C_k(X)$, which is complete since X is a k_R-space (see [6]). Therefore $f_\alpha \xrightarrow{k} f$ for some $f \in C(X)$. Now for each $\phi \in B_0(X)$ and $\varepsilon > 0$, there exists α_0 such that $p_\phi(f_\alpha - f_{\alpha'}) < \varepsilon$ for α, $\alpha' > \alpha_0$; that is, $\{\phi f_\alpha\}$ is u-Cauchy in $F(X)$. So $\phi f_\alpha \xrightarrow{u} g$ for some $g \in F(X)$ and, in particular, $\phi f_\alpha(x) \to g(x)$ pointwise for each $x \in X$. Since $f_\alpha \xrightarrow{k} f$, $f_\alpha \to f$ pointwise, so that $g(x) = \phi(x)f(x)$ for each $x \in X$. We conclude that $\phi f_\alpha \xrightarrow{u} \phi f$ for each $\phi \in B_0(X)$, and hence each $p_\phi(f_\alpha - f) \to 0$. Therefore $f_\alpha \to f$ in $C_\beta(X)$. ∎

Note that the sufficient condition in the above theorem is stronger than the necessary condition of Corollary 5.2. So we raise the following relevant question.

Problem *Can 5.2 be strengthened by obtaining X to be a k_R-space?*

The proofs of Theorems 5.1 and 5.3 can be modified to obtain the next two theorems.

Theorem 5.4 *For any space X, $C^*_{\sigma,u}(X)$ is closed in $C_{\sigma,u}(X)$.*

Theorem 5.5 *If X is a k_R-space, then $C_{\sigma,u}(X)$ is complete.*

From these two theorems, we immediately get the following result which improves Gulick's Theorem 4.10 in [4], in which X is assumed to be either locally compact or first countable.

Corollary 5.6 *If X is a k_R-space, then $C^*_{\sigma,u}(X)$ is complete.*

We turn now to the completeness of $C_\tau(X)$. Note that whenever X is locally almost σ-compact, $C_k(X) \leq C_\tau(X)$, and consequently the proof of Theorem 5.3 can be modified to obtain the following theorem.

Theorem 5.7 *If X is a locally almost σ-compact k_R-space, then $C_\tau(X)$ is complete.*

We point out that while a locally compact space is a k_R-space, a locally almost σ-compact space need not be a k_R-space. Example 3.10 provides an example of such a space. If X is locally compact, then Buck's proof of the completeness of $C^*_\beta(X)$ in [1] can be modified to show that $C^*_\tau(X)$ is complete, and as a consequence, $C^*_\tau(X)$ is closed in $C_\tau(X)$. This may not be true in the absence of local compactness.

The hypothesis that X be a k_R-space is not necessary in any one of Theorem 5.6, Corollary 5.7 and Theorem 5.8. This is because if X is almost σ-compact, then $C_\tau(X) = C_{\sigma,u}(X) = C_u(X)$; but both $C_u(X)$ and $C^*_u(X)$ are complete. On the other hand, Example 3.12 provides an example of a non-k_R-space for which neither $C^*_\beta(X)$ nor $C^*_{\sigma,u}(X)$ is complete. Also for this space, neither $C_\tau(X)$ nor $C^*_\tau(X)$ is complete.

6. Metrizability

The space $C_u(X)$ is of course always metrizable, and the space $C_k(X)$ is metrizable if and only if X is hemicompact (see [6], for example). As for the intermediate topologies, first Gulick established in [4] that $C^*_\beta(X)$ is metrizable if and only if X is compact. This immediately extends to $C_\beta(X)$.

Theorem 6.1 *The following are equivalent.*

(a) X is compact.

(b) $C_\beta(X)$ is metrizable.

(c) $C^*_\beta(X)$ is metrizable.

The metrizability of $C_\tau(X)$ is given by the next theorem.

Theorem 6.2 *Let X be a locally almost σ-compact space. Then the following are equivalent.*

(a) X contains a dense σ-compact subset.

(b) $C_\tau(X)$ is metrizable.

(c) $C_\tau(X)$ is first countable.

(d) $C^*_\tau(X)$ is first countable.

(e) $C^*_\tau(X)$ is metrizable.

Proof. The fact that (a) implies (b) follows from the fact that if X contains a dense σ-compact subset then $C_\tau(X) = C_u(X)$. Clearly (b) implies (c) and also (c) implies (d). The equivalence of (d) and (e) is because $C^*_\tau(X)$ is a locally convex Hausdorff space.

Finally we show that (e) implies (a). Let $\{V_{\phi_n,\varepsilon_n} n \in \mathbb{N}\}$ be a countable local base at f_0, where $V_{\phi_n,\varepsilon_n} = \{f \in C(X)\, p_{\phi_n}(f) < \varepsilon_n\}$, $\phi_n \in C_1(X)$ and $0 < \varepsilon_n < 1$ for each $n \in \mathbb{N}$. Also for each $n \in \mathbb{N}$, let S_n be a σ-compact set such that $\phi_n(X - \overline{S_n}) = 0$. We claim that $\overline{S} = X$, where $S = \cup_{n=1}^{\infty} S_n$. If not, then there exist $x_0 \in X - \overline{S}$ and continuous $g\, X \to [0,1]$ such that $g(x_0) = 1$ and $g = 0$ on $\overline{S}$. Now $p_{\phi_n}(g) = 0$ for all n, so that $g \in \cap_{n=1}^{\infty} V_{\phi_n,\varepsilon_n}$. But $\cap_{n=1}^{\infty} V_{\phi_n,\varepsilon_n} = \{f_0\}$, which is a contradiction. ∎

Finally let us mention that the metrizability of $C_{\sigma,u}(X)$ was studied in [5], and the characterization of when this space is metrizable is the same as in Theorem 6.2 for the space $C_\tau(X)$.

7. Separability

A space X is called ***submetrizable*** if there exist a metric space Y and a continuous bijection $h X \to Y$. We call a submetrizable space ***separably submetrizable*** if the metric space Y can be taken as separable. A submetrizable space X will be separably submetrizable whenever the density of X is less than or equal to $2^{\aleph_0}$.

The submetrizability of X plays a crucial role in determining the separability of $C(X)$ with various different topologies. For example, it is known that $C_k(X)$, $C_k^*(X)$ and $C_\beta^*(X)$ are all separable if and only if X is separably submetrizable (see [6], [10] or [11]). To determine the separability of $C_\beta(X)$ and $C_\tau(X)$ we need the following theorem found in [5].

Theorem 7.1 *Every pseudocompact submetrizable Tychonoff space is metrizable and hence compact.*

The spaces $C_u(X)$ and $C_u^*(X)$ are separable if and only if X is compact and metrizable. In the next theorem we see that is also true for $C_\beta(X)$.

Theorem 7.2 *For every space X, $C_\beta(X)$ is separable if and only if X is compact and metrizable.*

Proof. Because of Theorem 7.1, it suffices to show that if X is not pseudocompact then $C_\beta(X)$ is not separable. So suppose we have $A \subset X$ and $\phi \in B_0(X)$ as in Theorem 4.1. Then given any countable family $\{f_n \; n \in \mathbb{N}\} \subset C(X)$, define $f \; A \to \mathbb{R}$ by $f(x_n) = f_n(x_n) + \frac{1}{\phi(x_n)}$. There exists a continuous extension F of f to X. Then $p_\phi(F - f_n) = 1$ for each n, which shows that no countable subset of $C(X)$ can be dense in $C_\beta(X)$. ∎

To get the corresponding characterization for the separability of $C_\tau(X)$, we must assume an additional hypothesis on X.

Theorem 7.3 *Let X be a locally almost σ-compact space. Then the following are equivalent.*

(a) X is compact and metrizable.

(b) X is normal and $C_\tau(X)$ is separable.

(c) X is normal and $C_\tau^*(X)$ is separable.

Proof. We know that (a) implies (b) and (c). To show that (b) implies (a), we need only show that for a normal space X, if X is not pseudocompact then $C_\tau(X)$ is not separable. So choose $A \subset X$ and $\phi \in C_1(X)$ as in Theorem 4.2. Then given any countable family $\{f_n \; n \in \mathbb{N}\} \subset C(X)$, define $f \; A \to \mathbb{R}$ by

$f(x_n) = f_n(x_n) + 1$. Now the arguments similar to those in Theorem 7.2 show that $C_\tau(X)$ is not separable.

To show that (c) implies (a), suppose X is normal and $C_\tau^*(X)$ is separable. Then $C_k^*(X)$ is separable, so that X is submetrizable. It remains to show that X is pseudocompact. If not, then choose $A \subset X$ and $\phi \in C_1(X)$ as in Theorem 4.2. If $i\, A \to X$ denotes the inclusion map, then the induced map $i^*\, C^*(X) \to C^*(A)$ is a surjection. It can be easily shown that $i^*(V_{\phi,\varepsilon}) \subset V_\varepsilon$ where $V_{\phi,\varepsilon} = \{f \in C^*(X)\, p_\phi(f) < \varepsilon\}$, $V_\varepsilon = \{f \in C^*(X)\, \|f\|_\infty < \varepsilon\}$, $\|f\|_\infty = \sup\{|f(x)|\, x \in X\}$ and $0 < \varepsilon < 1$. Since $C_\tau^*(A) = C_u^*(A)$, it follows that i^* is continuous. So $C_u^*(A)$ is separable, and hence A is compact. This contradiction shows that indeed X is pseudocompact. ∎

We can establish a variation of Theorem 7.3 where the normality assumption on X is dropped by taking X to be locally compact.

Theorem 7.4 *Let X be a locally compact space. Then the following are equivalent.*

(a) X is compact and metrizable.

(b) $C_\tau(X)$ is separable.

(c) $C_\tau^*(X)$ is separable.

Proof. To show that (b) implies (c), suppose $C_\tau(X)$ is separable. Since X is locally compact, $C_\beta(X) \leq C_\tau(X)$, so that $C_\beta(X)$ is separable. Then by Theorem 7.2, X is compact and $C_\tau^*(X) = C_\tau(X)$.

Finally to show that (c) implies (a), suppose $C_\tau^*(X)$ is separable. Since X is submetrizable, we need to show that X is pseudocompact. Suppose not, so there exists an unbounded $f \in C(X)$. Then there are sequences $\{x_n\}$ in X and $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}$ such that for each n, $n < a_n < f(x_n) < b_n < a_{n+1}$. Since X is locally compact, each x_n has a compact neighborhood A_n which is contained in $f^{-1}((a_n, b_n))$. For each n, let $\phi_n\, X \to [0,1]$ be continuous so that $\phi_n(x_n) = 1$ and $\phi_n(X - A_n) = 0$. Then define $\phi\, X \to [0,1]$ by $\phi(x) = \phi_n(x)$ if $x \in A_n$ and $\phi(x) = 0$ otherwise. We see that $\phi \in C_1(X)$ since $\{A_n\, n \in \mathbb{N}\}$ is a discrete family in X. Now let $i\, \mathbb{N} \to X$ be defined by $i(n) = x_n$. Then the surjection $i^*\, C_\tau^*(X) \to C_\tau^*(\mathbb{N})$ is continuous as shown in Theorem 7.3, which leads to a contradiction. ∎

The separability of $C_{\sigma,u}(X)$ has been studied in [5], and this characterization is the same as that for $C_\tau(X)$ in Theorem 7.4 except that the local compactness is not needed.

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