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Krzysztof Banaszewski, Instytut Matematyki WSP, ul. Chodkiewicza 30, 85–064 Bydgoszcz, Poland

Algebraic Properties of $\mathcal{E}$–continuous Functions

1. Introduction.

Let $x \in \mathbb{R}$. A *path leading to* x is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and x is a point of bilateral accumulation of E_x. For $x \in \mathbb{R}$ let $\mathcal{E}(x)$ be a family of paths leading to x. A *system of paths* is a collection $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ such that each $E_x \in \mathcal{E}(x)$ for every $x \in \mathbb{R}$ (compare with [1]). Sometimes we shall simply refer to E_x as a "path."

We say that L_x (R_x) is a *left (right) path leading to* x if $L_x = E_x \cap (-\infty, x]$ $(R_x = E_x \cap x, \infty))$ for some path $E_x \in \mathcal{E}(x)$.

We shall only consider system of paths $\mathcal{E}$ having the property that if L_x is a left path leading to x and R_x is a right path leading to x then $L_x \cup R_x$ is an element of $\mathcal{E}(x)$, and we shall assume that $\mathbb{R} \in \mathcal{E}(x)$ for each $x \in \mathbb{R}$. We shall classify systems of paths according to the following scheme: a system of paths $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ will be said to be

– of *δ-type*, if $E_x \cap [x-\delta, x+\delta]$ contains a path in $\mathcal{E}(x)$ for every $E_x \in \mathcal{E}(x)$ and for every $\delta > 0$.

– of *σ-type*, if $\mathcal{E}$ is a δ-type system of paths, and for each triple of sequences of numbers $(a_n)_{n=1}^{\infty}$, $(x_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $b_{n+1} < a_n < x_n < b_n$, $(a_n < x_n < b_n < a_{n+1})$ $b_n \searrow x$ $(a_n \nearrow x)$ and for each left or right or bilateral paths $E_{x_n} \subset [a_n, b_n]$ leading to x_n for $n \in \mathbb{N}$, the set $\bigcup_{n=1}^{\infty} E_{x_n} \cup \{x\}$ contains a right path R_x (left path L_x) derived from an $E_x \in \mathcal{E}(x)$.

– of *c-type*, if $\mathcal{E}$ is a σ-system of paths and every Cantor set C_x such that x is a bilateral point of accumulation of C_x, belongs to $\mathcal{E}(x)$.

Such systems will be called shortly *δ-systems*, *σ-systems* and *c-systems*, respectively. We consider real functions of a real variable, unless otherwise explicitly stated.

Let X be a topological space, let $f : \mathbb{R} \to X$, and let $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ be a system of paths. We say that *function f is $\mathcal{E}$–continuous at x (f has a path at x)* if there exists a path $E_x \in \mathcal{E}(x)$ such that $f : E_x$ is continuous at x. If f is $\mathcal{E}$–continuous at every point x, then we say that f is *$\mathcal{E}$–continuous*.

We say that function f *has a left (right) path at* x if there exists a left (right) path $E_x \in \mathcal{E}(x)$ such that $f : E_x \cap (-\infty, x]$ ($f : E_x \cap x, \infty)$) is continuous at x. Let us settle some of the notation used in this article:

Const — the class of constant functions,

$\mathcal{C}$ — the class of continuous functions,

usc — the class of upper semicontinuous functions,

lsc — the class of lower semicontinuous functions,

$\mathcal{P}R$ — the class of all functions having a perfect road at each point of the domain,

$\mathcal{P}C$ — the class of peripherally continuous functions,

$\mathcal{D}$ — the class of Darboux functions,

$\mathcal{D}B_1$ — the class of Darboux functions of the first class of Baire,

Conn — the class of connectivity functions f for which it is true that for every connected subset G of $\mathbb{R}$ $f : G$ is a connected subset of $\mathbb{R}^2$,

$\mathcal{A}$ — the class of almost continuous functions f (in the sense of Stallings) such that for every open set $G \subset \mathbb{R}^2$ containing f there exists a continuous functions $g : \mathbb{R} \to \mathbb{R}$ contained entirely in G,

$\mathcal{F}$ — the class of functionally connected functions [1],

$\mathcal{M}$ — the class of functions f for which the following condition is satisfied: if x_0 is a right–hand sided (left–hand sided) point of discontinuity of f then $f(x_0) = 0$ and there is a sequence $x_n \searrow x_0$ ($x_n \nearrow x_0$) such that $f(x_n) = 0$.

Throughout we shall use the symbols $K^-(f, x)$ and $K^+(f, x)$ to denote the cluster sets from the left-hand side and right-hand side of the function f at the point x, respectively. By $Pr_x(A)$ and $Pr_y(A)$ we shall denote the x-projection and y-projection of a set $A \in \mathbb{R}^2$, respectively.

Notice that if $f \in \mathcal{M}$ then the set $\mathcal{W}$ of all points of discontinuity of f is nowhere dense and $f(x) = 0$ for each $x \in \overline{\mathcal{W}}$. Consequently f is a function of the first class of Baire, hence $\mathcal{M} \subseteq \mathcal{D} \cap \mathcal{B}_1$, since $\mathcal{P}R \cap \mathcal{B}_1 = \mathcal{P}C \cap \mathcal{B}_1 = \mathcal{D}B_1$ [1, 1, 1]. Let $\mathcal{X}$ be an class of real functions. The *maximal additive (multiplicative, latticelike) class for $\mathcal{X}$* we define as the class of all functions $f \in \mathcal{X}$ for which $f + g \in \mathcal{X}$ ($fg \in \mathcal{X}$, $\max(f, g) \in \mathcal{X}$ and $\min(f, g) \in \mathcal{X}$, respectively) whenever $g \in \mathcal{X}$. We denote these classes by $\mathcal{M}_a(\mathcal{X})$, $\mathcal{M}_m(\mathcal{X})$, and $\mathcal{M}_l(\mathcal{X})$, respectively. Finally, let us define

$$\mathcal{M}_{\max}(\mathcal{X}) = \{f \in \mathcal{X} : \text{if } g \in \mathcal{X}, \text{ then } \max(f,g) \in \mathcal{X}\},$$
$$\mathcal{M}_{\min}(\mathcal{X}) = \{f \in \mathcal{X} : \text{if } g \in \mathcal{X}, \text{ then } \min(f,g) \in \mathcal{X}\}.$$

Note that

$$\mathcal{M}_l(\mathcal{X}) = \mathcal{M}_{\max}(\mathcal{X}) \cap \mathcal{M}_{\min}(\mathcal{X}).$$

For real functions of a real variable we know that $\mathcal{M}_a(\mathcal{C}onn) = \mathcal{M}_a(\mathcal{F}) = \mathcal{M}_a(\mathcal{D}\cap\mathcal{B}_1) = \mathcal{C}$ [1, 1] and $\mathcal{M}_m(\mathcal{D}\cap\mathcal{B}_1) = \mathcal{M}$ [1], moreover $\mathcal{M}_a(\mathcal{D}) = \mathcal{M}_m(\mathcal{D}) = \mathcal{C}onst$ [1] and $\mathcal{M}_m(\mathcal{A}) = \mathcal{M}_m(\mathcal{C}onn) = \mathcal{M}_m(\mathcal{F}) = \mathcal{M}$, $\mathcal{M}_l(\mathcal{A}) = \mathcal{M}_l(\mathcal{C}onn) = \mathcal{M}_l(\mathcal{F}) = \mathcal{M}_a(\mathcal{A}) = \mathcal{C}$ [1].

It is obvious that for any system of paths $\mathcal{E}$ the class of all $\mathcal{E}$–continuous functions is contained in the class of peripherally continuous functions.

In the present article we shall prove that:

$$\mathcal{M}_a(\mathcal{P}R) = \mathcal{M}_a(\mathcal{P}C) = \mathcal{C},$$
$$\mathcal{M}_{\max}(\mathcal{P}R) = \mathcal{M}_{\max}(\mathcal{P}C) = \mathcal{C},$$
$$\mathcal{M}_{\min}(\mathcal{P}R) = \mathcal{M}_{\min}(\mathcal{P}C) = \mathcal{C},$$
$$\mathcal{M}_m(\mathcal{P}R) = \mathcal{M}_m(\mathcal{P}C) = \mathcal{M}.$$

In the third section we give examples of some systems of paths. In the next sections we shall consider representations of real function as sums, products, maximums and minimums, and pointwise limits of $\mathcal{E}$–continuous functions. Finally we characterize functions which can be represented as the maximum of two functions having a perfect road at each point of $\mathbb{R}$.

2. Some Basic Lemmas.

Let $\mathcal{E}$ be a system of paths and f a real function. It is obvious that if f has a left path L_x at x and a right path R_x at x then it has a path at x.

Lemma 2.1 *Let X be a topological space, $f : \mathbb{R} \to \mathbb{R}$, $f_1 : X \to \mathbb{R}$ continuous functions and $g : \mathbb{R} \to \mathbb{R}$, $g_1 : \mathbb{R} \to X$ $\mathcal{E}$-continuous functions. Then:*
i) $h = (f, g)$ is an $\mathcal{E}$-continuous function,
ii) $k = f_1 \circ g_1$ is an $\mathcal{E}$-continuous function.

Proof. Let $x \in \mathbb{R}$ and let E_x, F_x be paths leading to x such that $g : E_x$ and $g_1 : F_x$ are continuous at x. Note that $h : E_x = (f,g) : E_x = (f : E_x, g : E_x)$ and $k : F_x = (f_1 \circ g_1) : F_x = f_1 \circ (g_1 : F_x)$. Obviously $h : E_x$ and $k : F_x$ are continuous at x and therefore h and k are $\mathcal{E}$–continuous at x.

Corollary 2.2 *Let $\mathcal{X}$ be the class of all $\mathcal{E}$-continuous functions. Then*

$$\mathcal{C} \subset \mathcal{M}_a(\mathcal{X}) \cap \mathcal{M}_m(\mathcal{X}) \cap \mathcal{M}_l(\mathcal{X})$$

Proof. Suppose that $f \in \mathcal{X}$ and $g \in \mathcal{C}$. In view of Lemma 2.1, $h = (f, g)$ is an $\mathcal{E}$-continuous function. Since $k_1(x, y) = x + y$, $k_2(x, y) = xy$, $k_3(x, y) = \max(x, y)$ and $k_4(x, y) = \min(x, y)$ are continuous functions, so $f + g = k_1 \circ h$, $fg = k_2 \circ h$, $\max(f, g) = k_3 \circ h$ and $\min(f, g) = k_4 \circ h$ are $\mathcal{E}$-continuous functions.

Lemma 2.3 *Let $\mathcal{E}$ be a σ-system of paths, $f : \mathbb{R} \to \overline{\mathbb{R}}$ and let P be the set of points of $\mathcal{E}$-continuity of f. If the graph of $f : P$ is bilaterally dense in the graph of f then f is $\mathcal{E}$-continuous.*

Proof. Choose an $x \in \mathbb{R}$. We shall prove that f has a right path at x. Let (x_n) be a sequence of real numbers such that $x_n \searrow x$, $f(x_n) \to f(x)$ and $x_n \in P$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ choose a path E_{x_n} leading to x_n such that $f : E_{x_n}$ is continuous at x_n and note that there exists $\delta_{n,1} > 0$ such that if $: y - x_n :< \delta_{n,1}$ then $: f(y) - f(x_n) :< 1/n$ for $y \in E_{x_n}$. Let $\delta_n = \min(\delta_{n,1}, : x_n - x_{n-1} : /2, : x_n - x_{n+1} : /2)$. Since $\mathcal{E}$ is a σ-system there exists a path $F_n \subset \mathcal{E}(x)$ such that $F_n \subseteq E_{x_n} \cap (x_n - \delta_n, x_n + \delta_n)$ and there exist a right path $E_x \subseteq \bigcup_{n=1}^{\infty} F_n \cup \{x\}$. Observe that $f : E_x$ is continuous at x. In the same way we can prove that f has a left path at x.

Theorem 2.4 *If $\mathcal{E}$ is a σ-system of paths then the limit of a uniformly convergent sequence of $\mathcal{E}$-continuous functions is an $\mathcal{E}$-continuous function.*

Proof. Let $(f_n)_n^{\infty}$ be a sequence of $\mathcal{E}$-continuous functions which converges uniformly to function f. Choose any $x_0 \in \mathbb{R}$. We shall show that f has a right path at x_0. Let $m \in \mathbb{N}$. There exists $n_m \in \mathbb{N}$ such that $f_n - f < 1/(4m)$ for any $n \geq n_m$. Let E_m be an element of $\mathcal{E}(x_0)$ such that $: f_{n_m}(x) - f_{n_m}(x_0) :< 1/(4m)$ for every $x \in E_m$. Notice that there are two sequences of positive numbers (γ_m^1), (γ_m^2) such that $\gamma_{m+1}^2 < \gamma_m^1 < \gamma_m^2 < \gamma_{m-1}^1$, $\gamma_m^1 \to 0^+$ and the set $F_m = E_m \cap (x_0 + \gamma_m^1, x_0 + \gamma_m^2)$ is nonempty. Let $x_m \in F_m$ and $K_m \subseteq [x_0 + \gamma_m^1, x_0 + \gamma_m^2]$ be a path leading to x_m such that $: f_{n_m}(y) - f_{n_m}(x_m) :< 1/(4m)$ if $y \in K_m$. Because $\mathcal{E}$ is a σ-system then there exists a right path $R_{x_0} \subseteq \bigcup_{m=1}^{\infty} K_m \cup \{x_0\}$ leading to x_0. Indeed, for $\varepsilon > 0$ there exists an $m \in \mathbb{N}$ such that $1/m < \varepsilon$ and if $x \in R_{x_0}$ and $0 < x - x_0 < \gamma_m^2$ then $: f(x) - f(x_0) :\leq: f(x) - f_{n_m}(x) : + : f_{n_m}(x) - f_{n_m}(x_m) : + : f_{n_m}(x_m) - f_{n_m}(x_0) : + : f_{n_m}(x_0) - f(x_0) :< \varepsilon$. In a similar way we can prove that f has a left path leading to x_0.

3. Examples.

Many classes of functions can be treated as families of all $\mathcal{E}$-continuous functions, where $\mathcal{E}$ is some system of paths.

(a) The class of continuous (approximately continuous) functions. Let $\mathcal{T}$ be the Euclidean (the density) topology on $\mathbb{R}$. Put $\mathcal{E}(x) = \{A : \exists_{V\in\mathcal{T}}\ x \in V \subset A\}$ and $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$.
(b) The class of almost continuous functions (in the sense of Husain [1]). Put $\mathcal{E}(x) = \{A : x \in A \text{ and } \exists_{\delta>0}\ (x-\delta, x+\delta) \subset \overline{A}\}$ and $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$.
(c) The class of bilaterally quasi-continuous functions. Put $\mathcal{E}(x) = \{A : x \in A \text{ and } \forall_{\delta>0}(\exists_{a,b,c,d}\ x-\delta < a < b < x < c < d < x+\delta \text{ and } (a,b)\cup(c,d) \subset A)\}$ and $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$.
(d) The class ($\mathcal{PR}$) of functions having a perfect road everywhere, i.e. functions f such that for each $x \in \mathbb{R}$ we can find a perfect set E_x such that x is a point of its bilateral accumulation and $f : E_x$ is continuous at x. Let $\mathcal{E}(x)$ be the set of all perfect sets such that x is a point of bilateral accumulation and $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$
(e) The class ($\mathcal{PC}$) of peripherally continuous functions [1], i.e. functions f such that for each $x \in \mathbb{R}$ there exist sequences $x_n \searrow x$, $y_n \nearrow x$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(x)$. Let $\mathcal{E}(x)$ be the set of all sets containing x such that x is a point of bilateral accumulation and $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$
(f) The class $C(m)$ of functions having the following property:
$\forall_{x\in\mathbb{R}}\forall_{\delta>0}\exists_{P\subset\mathbb{R}}\ \operatorname{card}(P\cap(x, x+\delta)) \geq m$, $\operatorname{card}(P\cap(x-\delta, x)) \geq m$ and $f : P$ is continuous at x, where m is a fixed infinite cardinal number less than or equal to the continuum. Note that this class is equal to $\mathcal{PC}$ if $m = \omega_0$. If $m > \omega_0$ then $\mathcal{PR} \subset C(m) \subset \mathcal{PC}$ and $C(m) \neq \mathcal{PC}$ (consider the characteristic function of rationals as example) and $C(m) \neq \mathcal{PR}$ (For example, consider the characteristic function of a Bernstein set).
(g) The class of *rational–irrational* functions (RQ). Let

$$E_x^\varepsilon = \begin{cases} Q\cap(x-\varepsilon, x+\varepsilon) & \text{if } x \in Q \\ (\mathbb{R}\setminus Q)\cap(x-\varepsilon, x+\varepsilon) & \text{if } x \in \mathbb{R}\setminus Q \end{cases}$$

for each $\varepsilon > 0$, $\mathcal{E}(x) = \{E_x^\varepsilon : \varepsilon > 0\}$ and $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ be a system of paths. Then $f : \mathbb{R} \to \mathbb{R}$ is rational–irrational function iff $f : E_x^\varepsilon$ is continuous at x for every $x \in \mathbb{R}$ and $\varepsilon > 0$.

It is easy to see that in all the above cases a function f belongs to the mentioned class iff f is $\mathcal{E}$–continuous. Note that the path systems in (a), (b) and (g) are not σ-systems and the systems of paths in (d), (e), (f) are c-systems.

Example 3.1 *Let $\mathcal{E}$ be a c-system of paths and $I = [0, 1]$. There exists an $\mathcal{E}$-continuous function $f : I \to I$ such that f is discontinuous everywhere, f is a bijection and $f = f^{-1}$.*

Let $(I_n)_n^\infty$ be a sequence of all open intervals contained in I with rational endpoints. In each I_n we choose a Cantor set C_n such that $C_m \cap C_n = \emptyset$ for $m \neq n$.

Such a sequence $(C_n)_n^\infty$ exists since for each $n \in \mathbb{N}$ the set $I_n \setminus \bigcup_{m=1}^{n-1} C_m$ is a G_δ uncountable set and so it contains a Cantor set C_n (see [1], page 387). Put

$$f(x) = \begin{cases} x & \text{if } x \in \bigcup_{n=1}^\infty C_n \text{ or } 1-x \in \bigcup_{n=1}^\infty C_n, \\ 1-x & \text{otherwise.} \end{cases}$$

It is easy to verify that f is satisfy the requirements.

4. Operations.

In the present section we shall consider the maximal additive (multiplicative, latticelike) class for the class of $\mathcal{E}$–continuous functions for a σ-system or a c-system of paths $\mathcal{E}$.

Theorem 4.1 *If $\mathcal{E}$ is a σ-system of paths and $\mathcal{X}$ is the class of all $\mathcal{E}$–continuous functions then*

$$\mathcal{M}_a(\mathcal{X}) = \mathcal{C}.$$

Proof. In view of Corollary 2.2 we need only to prove that $\mathcal{M}_a(\mathcal{X}) \subseteq \mathcal{C}$. Let f be an $\mathcal{E}$–continuous function and suppose that f is not continuous at x_0 from the right. We shall consider two cases:
i) There exists a $y \in K^+(f, x_0) \setminus \{f(x_0), -\infty, +\infty\}$. Let $c = f(x_0) - y$ and put

$$g(x) = \begin{cases} c - f(x) & \text{for } x \le x_0 \\ -f(x) & \text{for } x > x_0. \end{cases}$$

Because the set $g : (-\infty, x_0) \cup (x_0, \infty)$ is bilaterally dense in g then by Lemma 2.3 the function g is $\mathcal{E}$–continuous. Notice that $(f+g)(x) = c$ for $x \le x_0$ and $(f+g)(x) = 0$ for $x > x_0$, so then function $f+g$ is not $\mathcal{E}$–continuous at x_0.
ii) If $K^+(f, x_0) \subseteq \{f(x_0), -\infty, +\infty\}$ then let

$$g(x) = \begin{cases} 0 & \text{for } x \le x_0 \\ e^{-:f(x):} & \text{for } x > x_0. \end{cases}$$

Since $h(y) = e^{-:y:}$ is continuous, from Lemma 2.1 it follows that the function g is $\mathcal{E}$–continuous at x for every $x \in (-\infty, x_0) \cup (x_0, +\infty)$. Because $0 \in K^+(g, x_0)$, $g : (-\infty, x_0) \cup (x_0, +\infty)$ is bilaterally dense in g and by Lemma 2.3 the function g is $\mathcal{E}$–continuous. Notice that $(f+g)(x_0) = f(x_0)$ and $(f+g)(x) = f(x) + e^{-:f(x):}$ for $x > x_0$. We shall prove that $(f+g)$ is not peripherally continuous. Let $(x_n)_n$ be a sequence such that $x_n \searrow x_0$. Two cases may occur:
a) For every subsequence $(x_{n_m})_m$ of the sequence $(x_n)_n$ we have $\lim_{m\to\infty} : f(x_{n_m}) := \infty$. Then $\lim_{m\to\infty} : f + g : (x_{n_m}) = \infty$.
b) Suppose that $(x_{n_m})_m$ is a subsequence of the sequence $(x_n)_n$ such that $f(x_{n_m})$ is bounded. Therefore $f(x_{n_m}) \to f(x_0)$ and $\lim_{m\to\infty}(f+g)(x_{n_m}) = f(x_0) + e^{-:f(x_0):}$. Consequently, $(f+g)(x_0) \notin K^+(f+g, x_0)$, which completes the proof.

Theorem 4.2 *If $\mathcal{E}$ is a σ-system of paths and $\mathcal{X}$ is the class of all $\mathcal{E}$–continuous functions then*

$$\mathcal{M}_m(\mathcal{X}) = \mathcal{X} \cap \mathcal{M}.$$

Proof. First we verify that $\mathcal{X} \cap \mathcal{M} \subseteq \mathcal{M}_m(\mathcal{X})$. Let $f \in \mathcal{M} \cap \mathcal{X}$, let g be an $\mathcal{E}$–continuous function and let x_0 be arbitrary. Suppose that R_{x_0} is a right path leading to x_0 such that $g : R_{x_0}$ is continuous at x_0. We shall prove that the function fg has a right path at x_0. We shall consider two cases:

i) The function f is continuous at x_0 from the right. Then

$$fg : R_{x_0} = (f : R_{x_0})(g : R_{x_0})$$

is continuous at x_0.

ii) The function f is not continuous at x_0 from the right. Then there exists a sequence $x_n \searrow x_0$ of points at which f is right-hand sided or left-hand sided continuous and $f(x_n) = 0$ [1]. Assume that $(x_n)_{n=1}^{\infty}$ is a sequence of points of right-hand sided continuity of f. Let $n \in \mathbb{N}$, $\delta_{n,1} =: x_n - x_{n-1} :$, $\delta_{n,2} =: x_n - x_{n+1} :$, $\varepsilon_n = 1/n$ if $g(x_n) = 0$ and $\varepsilon_n = \min(1/n, 1/(n : g(x_n) :))$ otherwise. There exists a $\delta_{n,3}$ such that if $0 < x - x_n < \delta_{n,3}$ then $: f(x) :< \varepsilon_n$. Let $\delta_n = \min(\delta_{n,1}/2, \delta_{n,2}/2, \delta_{n,3})$. Denote by Q_n a right path leading to x_n such that $: g(x) - g(x_n) :< 1/n$ for $x \in Q_n$. Then there is a right path K_n leading to x_n contained in the set $Q_n \cap [x_n, x_n + \delta_n]$ and there exists a right path $K_0 \subseteq \bigcup_{n=1}^{\infty} K_n \cup \{x_0\}$ leading to x_0.

We shall prove that function $fg : K_0$ is continuous at x_0. Let $(y_m)_m^{\infty}$ be a sequence of points such that $y_m \in K_0 \setminus \{x_0\}$ for each $m \in \mathbb{N}$ and $y_m \searrow x_0$. Note that for any $m \in \mathbb{N}$ there exists an $n_m \in \mathbb{N}$ such that $y_m \in K_{n_m}$ and $m \to \infty$ implies $n_m \to \infty$. Then $: (fg)(y_m) :< 2/n_m$ and hence the sequence $(fg)(y_m)$ converges to $(fg)(x_0)$. In the same way we can prove that (fg) has a left path at x_0.

Now we shall prove that $\mathcal{M}_m(\mathcal{X}) \subseteq \mathcal{X} \cap \mathcal{M}$. Assume that $f \in \mathcal{X} \setminus \mathcal{M}$. Then there exists a point x_0 of right-hand sided (or left-hand sided) discontinuity of f such that one from the following conditions is satisfied:

i) There exists a $\delta > 0$ for which $f(x) \neq 0$ for $x \in [(x_0, x_0 + \delta]$ and there is a $y \in K^+(f, x_0) \setminus \{f(x_0), 0\}$. Let $c = 1/y$ if y is finite and $c = 0$ otherwise. Put

$$g(x) = \begin{cases} c & \text{if } x \le x_0 \\ 1/f(x) & \text{if } x \in [(x_0, x_0 + \delta] \\ 1/f(x_0 + \delta) & \text{if } x > x_0 + \delta. \end{cases}$$

Because g is $\mathcal{E}$–continuous at each point of the set $(-\infty, x_0) \cup (x_0, \infty)$ and $g(x_0) \in K^+(g, x_0)$, so by Lemma 2.3 g is $\mathcal{E}$–continuous. Notice that

$(fg)(x) = 1$ for all $x \in (x_0, x_0 + \delta)$ and $(fg)(x_0) \neq 1$. Hence $(fg) \notin \mathcal{X}$ and consequently $f \notin \mathcal{M}_m(\mathcal{X})$.

ii) The function f fulfills one of the following two conditions:

a) $K^+(f, x_0) = \{f(x_0), 0\}$ and there exists $\delta > 0$ such that $f(x) \neq 0$ for $x \in [(x_0, x_0 + \delta]$ or

b) $f(x_0) \neq 0$ and there exists a sequence (x_n) such that $x_n \searrow x_0$ and $f(x_n) = 0$.

Then let

$$g(x) = \begin{cases} f(x_0) & \text{if } x \leq x_0 \\ f(x_0) - f(x) & \text{if } x > x_0. \end{cases}$$

By Lemma 2.1 and Theorem 4.1 the function $g : (-\infty, x_0) \cup (x_0, \infty)$ is $\mathcal{E}$–continuous. Because $0 \in K^+(f, x_0)$, the set $g : (-\infty, x_0) \cup (x_0, \infty)$ is bilaterally dense in g and, by Lemma 2.3, g is $\mathcal{E}$–continuous. Notice that $(fg)(x_0) = f^2(x_0) > 0$. We shall prove that $fg \notin \mathcal{PC}$. Let $(y_n)_{n=1}^{\infty}$ be a sequence of points convergent to x_0 from the right. If there is a subsequence $(y_{n_m})_{m=1}^{\infty}$ such that $f(y_{n_m})$ is convergent to some finite real number z then $\lim_{m\to\infty}(fg)(y_{n_m}) = f(x_0)z - z^2 \neq f^2(x_0)$ for each $z \in \mathbb{R}$. If $\lim_{m\to\infty} : f(y_{n_m}) := \infty$ then $\lim_{m\to\infty} fg(y_{n_m}) = -\infty$. Hence $fg \notin \mathcal{PC}$ and $fg \notin \mathcal{X}$. This implies that $f \notin \mathcal{M}_m(\mathcal{X})$.

If the function f is not continuous at x_0 from the left then the proof is similar.

Corollary 4.3 *Since $\mathcal{M} \subset \mathcal{P}R \subset \mathcal{PC}$, we have*

$$\mathcal{M}_m(\mathcal{P}R) = \mathcal{M}_m(\mathcal{PC}) = \mathcal{M}.$$

Theorem 4.4 *Let $\mathcal{E}$ be a c-system and $\mathcal{X}$ the class of all $\mathcal{E}$–continuous functions. Then the following equalities hold:*

$$\mathcal{M}_{\max}(\mathcal{X}) = \mathcal{M}_{\min}(\mathcal{X}) = \mathcal{C}.$$

Proof. We shall prove that $\mathcal{M}_{\max}(\mathcal{X}) = \mathcal{C}$ (the proof that $\mathcal{M}_{\min}(\mathcal{X}) = \mathcal{C}$ is similar). In Corollary 2.2 we showed that $\mathcal{C} \subseteq \mathcal{M}_{\max}(\mathcal{X})$. Now we shall prove the opposite inclusion. We shall consider two cases.

1. *[i)] $f \notin usc$. Then there exists a point x_0 such that*

$$f(x_0) < \limsup_{x\to x_0^+} f(x) \quad \textit{or} \quad f(x_0) < \limsup_{x\to x_0^-} f(x).$$

We shall deal only with the first case.

a) *If there exists $y \in K^+(f, x_0)$ such that $f(x_0) < y < \infty$, then let $d = (f(x_0) + y)/2$ and put*

$$g(x) = \begin{cases} f(x_0) & \text{if } x \le x_0 \\ 2d - f(x) & \text{if } x > x_0. \end{cases}$$

By Lemma 2.1 and Theorem 4.1 the functions $g : (x_0, \infty)$ and $g : (-\infty, x_0]$ are $\mathcal{E}$-continuous. Because $f(x_0) \in K^+(g, x_0)$, the set $g : (-\infty, x_0) \cup (x_0, \infty)$ is dense in g and by Lemma 2.3 g is $\mathcal{E}$-continuous. Moreover, $\max(f, g)(x_0) = f(x_0) < d$ and $\max(f, g)(x) \ge d$ for $x > x_0$, so $\max(f, g) \notin \mathcal{PC}$ and consequently $f \notin \mathcal{M}_{\max}(\mathcal{X})$.

b) *Otherwise $+\infty \in K^+(f, x_0)$. Put*

$$g(x) = \begin{cases} f(x) & \text{if } x \le x_0 \\ f(x_0) + e^{-f(x)} & \text{for } x > x_0. \end{cases}$$

Notice that in view of Lemma 2.1 and Theorem 4.1 the function

$$g\,|((-\infty, x_0) \cup (x_0, \infty))$$

is $\mathcal{E}$-continuous and the set $g : ((-\infty, x_0) \cup (x_0, \infty))$ is bilaterally dense in g. So by Lemma 2.3 g is $\mathcal{E}$-continuous. Observe that $\max(f, g) \notin \mathcal{PC}$. Indeed, let $x_n \searrow x_0$. If $\limsup_{n\to\infty} f(x_n) = +\infty$ then

$$\limsup_{n\to\infty} \max(f, g)(x_n) \ge \limsup_{n\to\infty} f(x_n) = \infty.$$

Otherwise

$$\limsup_{n\to\infty} \max(f, g)(x_n) \ge \limsup_{n\to\infty} g(x_n) \ge$$

$$f(x_0) + e^{-\limsup_{n\to\infty} f(x_n)} > f(x_0) = \max(f, g)(x_0).$$

Consequently, $f \notin \mathcal{M}_{\max}(\mathcal{X})$.

ii) *$f \in usc$. Assume that f is discontinuous at x_0 from the right. Then there is a point $y \in K^+(f, x_0)$ such that $y < f(x_0)$. Let $d = (f(x_0) + y)/2$ if y is finite and $d = f(x_0) - 1$ otherwise. Since $f \in usc$, $G = \{x : f(x) < d\} \cap (x_0, \infty)$ is a nonempty open set. Let $(I_n)_{n=1}^{\infty}$ be a sequence of all open components of G. Let $(I_{n,k})_{n,k=1}^{\infty}$ be a sequence of all open intervals with rational endpoint contained in I_n. In each $I_{n,k}$ we choose a Cantor set $C_{n,k}$ such that $C_{n,k} \cap C_{m,p} = \emptyset$ for $(n, m) \ne (k, p)$. Let us define $C = (-\infty, x_0] \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n,k}$ and*

$$g(x) = \begin{cases} d & \text{if } x \in C \\ f(x_0) + 1 & \text{otherwise.} \end{cases}$$

We shall verify that g is $\mathcal{E}$–continuous at each point. It is obvious at each point $x \in C$. Let $x \in \mathbb{R} \setminus C$ and $U_n = (x + 1/(n+1), x + 1/n)$ for $n \in \mathbb{N}$. Because $U_n \setminus C$ is a G_δ uncountable set, we can choose a Cantor set $K_n \subset U_n \setminus C$. Hence the set $R_x = \bigcup_{n=1}^{\infty} K_n \cup \{x\}$ is a right path leading to x and $g : R_x$ is continuous. Notice that $\max(f, g)(x_0) = f(x_0)$ *and* $\max(f, g)(x) = d < f(x_0)$ *or* $\max(f, g)(x) \geq f(x_0) + 1$ *for $x \geq x_0$. This implies that* $\max(f, g) \notin \mathcal{PC}$ *and therefore $f \notin \mathcal{M}_{\max}(\mathcal{X})$.*

Remark 4.1 *Note that for the system of rational-irrational paths defined in example (g) of section 3 the conclusions of each of Theorems 4.1–4.3 fail to hold and, thus, the assumptions that E is a σ-system is important.*

5. Representations.

Throughout whose this section $\mathcal{E}$ denotes some fixed c–system. Let $(I_k)_{k=1}^{\infty}$ be a sequence of open intervals endpoints are rationals. Let $(q_n)_{n=1}^{\infty}$ be an enumeration of rationals different from zero. In the proofs we shall use the fact that in each interval I_k we can choose a sequence $(C_{k,n})_{n=1}^{\infty}$ of Cantor sets in I_k such that $C_{k,n} \cap C_{m,p} = \emptyset$ for $(k, n) \neq (m, p)$ (cf. Example 3.1).

Theorem 5.1 *For any function $f : \mathbb{R} \to \mathbb{R}$ we can find $\mathcal{E}$–continuous functions g, h such that $f = g + h$. Moreover, if f is measurable (Baire class α for $\alpha \geq 2$) we can find such a representation that g and h are also measurable (Baire class α).*

Proof. Put

$$g(x) = \begin{cases} q_n & \text{if } x \in C_{k,2n},\ k,\ n \in \mathbb{N} \\ f(x) - q_n & \text{if } x \in C_{k,2n+1},\ k,\ n \in \mathbb{N} \\ f(x) & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} f(x) - q_n & \text{if } x \in C_{k,2n},\ k,\ n \in \mathbb{N} \\ q_n & \text{if } x \in C_{k,2n+1},\ k,\ n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then g is $\mathcal{E}$–continuous at every point of bilateral accumulation of $C_{k,2n}$, $k, n \in \mathbb{N}$, so by Lemma 2.3, g is $\mathcal{E}$–continuous. Analogously, h is $\mathcal{E}$–continuous. The second part of the theorem follows from the following equality:

$$g = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} g : C_{k,2n} \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} g : C_{k,2n+1} \cup f : (\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{k,n})$$

and that remark each of component functions is in this union of the same Baire classes f and they are defined on sets of the second Borel class. Analogously, if f is measurable then g is measurable too. The similar arguments work for g.

Theorem 5.2 *Let $f : \mathbb{R} \to \mathbb{R}$. Then there exist $\mathcal{E}$-continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = gh$. If f is measurable (Baire class α for $\alpha \geq 2$) we can require g and h to be measurable (Baire class α), too.*

Proof. Put

$$g(x) = \begin{cases} q_n & \text{if } x \in C_{k,2n},\ k, n \in \mathbb{N} \\ f(x)/q_n & \text{if } x \in C_{k,2n+1},\ k, n \in \mathbb{N} \\ f(x) & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} f(x)/q_n & \text{if } x \in C_{k,2n},\ k, n \in \mathbb{N} \\ q_n & \text{if } x \in C_{k,2n+1},\ k, n \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

Proceeding as in the proof of the previous theorem we get that g and h are $\mathcal{E}$–continuous (measurable and Baire class α provided that f is).

Theorem 5.3 *Every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as*

$$f = \min(\max(f_1, f_2), \max(f_3, f_4))$$

where f_1, f_2, f_3, f_4 are $\mathcal{E}$–continuous functions.

Proof. Put

$$f_i(x) = \begin{cases} q_n & \text{if } x \in C_{k,4n-i+1}, \quad k, n \in \mathbb{N} \\ f(x) & \text{otherwise} \end{cases}$$

and $D_i = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} C_{k,4n-i+1}$ for $i = 1, 2, 3, 4$. As above, f_i are $\mathcal{E}$–continuous. We shall verify that

$$f = \min(\max(f_1, f_2), \max(f_3, f_4)). \tag{1}$$

a) If $x \notin \bigcup_{i=1}^{4} D_i$, then $f_1(x) = f_2(x) = f_3(x) = f_4(x) = f(x)$, so (1) is satisfied.
b) If $x \in D_1$, then $x \notin \bigcup_{i=2}^{4} D_i$, so $f_2(x) = f_3(x) = f_4(x) = f(x)$. Then either $f_1(x) \leq f(x)$ and hence $\max(f_1, f_2)(x) = f(x)$, so

$$f(x) = \min \max(f_1(x), f_2(x)), \max(f_3(x), f_4(x))]$$

or $f_1(x) > f(x)$ whence

$$f(x) = \min[\max(f_1(x), f_2(x)), \max(f_3(x), f_4(x))].$$

c) If $x \in D_2 \cup D_3 \cup D_4$ then we proceed analogously.

Note 5.1 *If* $(-f) = \min(\max(f_1, f_2), \max(f_3, f_4))$ *then*

$$f = \max(\min(-f_1, -f_2), \min(-f_3, -f_4)),$$

so max *and* min *can be interchanged.*

Theorem 5.4 *Every real function of real variable is a pointwise limit of $\mathcal{E}$-continuous functions.*

Proof. Represent each $C_{k,n}$ as the union $\bigcup_{\alpha<c} C_{k,n,\alpha}$ of pairwise disjoint perfect sets (c-denotes the cardinality of $\mathbb{R}$) [1]. Let $(x_\alpha)_{\alpha<c}$ be a transfinite sequence of all reals. Put

$$D_{n,\alpha} = \bigcup_{k=1}^{\infty} C_{k,n,\alpha}$$

and

$$f_n(x) = \begin{cases} x_\alpha & \text{if } x \in D_{n,\alpha}\ ,\ \alpha < c \\ f(x) & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. Then f_n is $\mathcal{E}$-continuous. We shall show that

$$f(x) = \lim_{n\to\infty} f_n(x). \tag{2}$$

Choose an $x \in \mathbb{R}$. Then either $x \notin \bigcup_{n=1}^{\infty} \bigcup_{\alpha<c} D_{n,\alpha}$, $f_n(x) = f(x)$ for each $n \in \mathbb{N}$ and (2) holds or $x \in D_{n_0,\alpha}$ for some $n_0 \in \mathbb{N}$, $\alpha < c$, whence $x \notin D_{n,\alpha}$ for $n > n_0$, $\alpha < c$, so $f_n(x) = f(x)$ for $n > n_0$, which completes the proof.

Theorem 5.5 *Let $f : \mathbb{R} \to \mathbb{R}$. Then the following conditions are equivalent:*
(a) there exist Darboux functions g, h having a perfect road everywhere such that

$$f = \max(g, h),$$

(b) there exist functions g, h having a perfect road everywhere such that

$$f = \max(g, h),$$

(c) for each $x \in \mathbb{R}$ there exist perfect sets R, L such that x is both a point of accumulation of R from the right and a point of accumulation L from the left, for which the limits

$$\lim_{\substack{z\to x^+ \\ z\in R\setminus\{x\}}} f(z) \text{ and } \lim_{\substack{z\to x^- \\ z\in L\setminus\{x\}}} f(z)$$

exist (maybe infinite), and

$$f(x) \le \min\Big(\lim_{\substack{z\to x^+ \\ z\in R\setminus\{x\}}} f(z), \lim_{\substack{z\to x^- \\ z\in L\setminus\{x\}}} f(z)\Big).$$

(We shall say that R (L) is an upper perfect road of f at x from the right (from the left)).

Proof. $(a) \Rightarrow (b)$ Obvious.
$(b) \Rightarrow (c)$ Choose an $x \in \mathbb{R}$. Let

$$y_g = \limsup_{z \to x+} g(z)$$

and

$$y_h = \limsup_{z \to x+} h(z).$$

By an argument like the one used in Lemma 2.3 the functions

$$g_1(z) = \begin{cases} y_g & \text{if } z = x \\ g(z) & \text{otherwise} \end{cases}$$

and

$$h_1(z) = \begin{cases} y_h & \text{if } z = x \\ h(z) & \text{otherwise} \end{cases}$$

have a right perfect road everywhere. Thus there are perfect sets R_g, R_h such that x is a point of accumulation of both R_g and R_h from the right and

$$\lim_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} g(z) = y_g$$

$$\lim_{\substack{z \to x+ \\ z \in R_h \setminus \{x\}}} h(z) = y_h.$$

We can assume that $y_h \leq y_g$. Then

$$\liminf_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} \max(g(z), h(z)) \geq \liminf_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} g(z) = y_g,$$

$$\limsup_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} \max(g(z), h(z)) = \max(\lim_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} g(z), \limsup_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} h(z)) \leq$$

$$\leq \max(y_g, \limsup_{\substack{z \to x+ \\ z \in R_g \setminus \{x\}}} h(z)) \leq \max(y_g, y_h) = y_g.$$

Therefore R_g is an upper perfect road at x from the right. Finding an upper perfect road of f at x from the left is analogous.
$(c) \Rightarrow (a)$ Let $\mathcal{S} = \{S_n\}_{n=1}^{\infty}$ be an enumeration of all open intervals in $\mathbb{R}^2$ of the form $(a_1, a_2) \times (a_3, +\infty)$, where a_1, a_2, a_3 are rationals such that there exists an nonempty perfect set $P \subset Pr_x(f \cap S_n)$. By Lemma 2 of [1] we can find a sequence of pairwise disjoint, nonvoid perfect sets $\{P_n\}_{n=1}^{\infty}$ such that $P_{2n}, P_{2n-1} \subset Pr_x(S_n \cap f)$. For each $n \in \mathbb{N}$ we define a family of pairwise disjoint,

nonvoid perfect sets $\{P_{n,\alpha}\}_{\alpha<c}$ such that $P_n = \bigcup_{\alpha<c} P_{n,\alpha}$ (cf. Theorem 5.4). Put $a_n = \inf Pr_y(S_n)$ for each $n \in \mathbb{N}$ and let $\{x_{n,\alpha}\}_{\alpha<c}$ be a transfinite sequence not greater than a_n. Define

$$g(x) = \begin{cases} x_{n,\alpha} & \text{if } x \in P_{2n,\alpha},\ n \in \mathbb{N},\ \alpha < c \\ f(x) & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} x_{n,\alpha} & \text{if } x \in P_{2n+1,\alpha},\ n \in \mathbb{N},\ \alpha < c \\ f(x) & \text{otherwise.} \end{cases}$$

Then $\max(g,h) = f$, because
i) if $x \notin \bigcup_{n=1}^{\infty} P_n$ then $g(x) = h(x) = f(x)$,
ii) if $x \in P_{2n}$ for some $n \in \mathbb{N}$ then $g(x) \leq a_n$ and $h(x) = f(x) > a_n$,
iii) if $x \in P_{2n+1}$ for some $n \in \mathbb{N}$ then $g(x) = f(x) > a_n$ and $h(x) \leq a_n$.

Choose an $x \in \mathbb{R}$. We prove now that the function g has a perfect road at x. Let $(x_m)_{m=1}^{\infty}$, $(y_m)_{m=1}^{\infty}$ be sequences of real numbers such that $x_m \searrow x$, $f(x_m) > y_m$ and $y_m \to g(x)$. For $m = 2, 3, \ldots$ let n_m be such that

$$S_{n_m} \subseteq ((x_{m+1} + x_m)/2, (x_m + x_{m-1})/2) \times (y_m, +\infty)$$

and let $\alpha_m < c$ be such that $x_{n_m,\alpha_m} = y_m$. (Such an n_m exists since f has an upper perfect road at x_m.) Then the set $R = \bigcup_{m=1}^{\infty} P_{2n_m,\alpha_m} \cup \{x\}$ is perfect, x is its left point of accumulation from the right and

$$\lim_{\substack{z \to x+ \\ z \in R\setminus\{x\}}} g(z) = \lim_{m\to\infty} y_m = g(x).$$

In a similar way we can prove that g has a perfect road at x from the left.

Finally we show that g is a Darboux function. Let $a < b$ and $g(a) < \lambda < g(b)$. Since f has an upper perfect road at b from the left, there exists a perfect set $P \subseteq (a,b)$ such that $f(x) > \lambda$ for $x \in P$. Then $S_n \subseteq (a,b) \times (\lambda, \infty)$ for some $n \in \mathbb{N}$ and $\lambda = x_{n,\alpha}$ for some $\alpha < c$, so for each $x \in P_{2n,\alpha}$ we have $a < x < b$ and $g(x) = \lambda$.

The proof that h is a Darboux function having a perfect road everywhere is similar.

Analogously we can prove the following

Theorem 5.6 *Let $\mathcal{E}$ be a σ-system such that if $\{E_n : n \in \mathbb{N}\}$ is a collection of paths then there is a family $\{P_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty paths such that $P_n \subseteq E_n$ for $n \in \mathbb{N}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then the following conditions are equivalent:*
(a) *there exist $\mathcal{E}$-continuous functions g, h such that*

$$f = \max(g,h),$$

(b) for each $x \in \mathbb{R}$ there exist a right path R leading to x and a left path L leading to x, for which the limits

$$\lim_{\substack{z \to x+ \\ z \in R \setminus \{x\}}} f(z) \text{ and } \lim_{\substack{z \to x- \\ z \in L \setminus \{x\}}} f(z)$$

exist (maybe infinite), and

$$f(x) \leq \min\Big(\lim_{\substack{z \to x+ \\ z \in R \setminus \{x\}}} f(z), \lim_{\substack{z \to x- \\ z \in L \setminus \{x\}}} f(z)\Big).$$

Remark 5.1 *The classes $\mathcal{PC}$ and $C(m)$ for $\omega_0 < m \leq \omega_1$ fulfill the conditions of Theorem 5.6.*

Remark 5.2 *The assumption that $\mathcal{E}$ is a c-system is important in all theorems in this section. Consider, for example, the system of all open intervals.*

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