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On ω-limit Sets of Triangular Maps

1. Introduction

A number of papers, some dating back to the Sixties (see, e.g., [Sh]), deal with the ω-limit sets of continuous self-maps of the interval. Recently a full characterization of such sets has been found. As established in [ABCP] and [BS] a non-void closed subset M of $I = [0,1]$ is an ω-limit set for some continuous function $f : I \mapsto I$ if and only if M is nowhere dense or a union of finitely many nondegenerate closed intervals. The structure of ω-limit sets for some other classes of functions $I \mapsto I$ is studied in [BCP].

To characterize the closed sets which can be ω-limit sets for continuous maps from E^k into E^k is a difficult open problem. (Here E is the set of real numbers.) Only partial results are known (see [C]).

A natural approach to this open problem is to study ω-limit sets in the dimension two and consider only continuous maps of some special form. Triangular maps are a good example.

A map $F : I^2 \mapsto I^2$ is called triangular if $F(x,y) = (f(x), g(x,y))$, i.e. if the first coordinate of the image of a point depends only on the first coordinate of that point. The map F is continuous if and only if $f : I \mapsto I$ and $g : I^2 \mapsto I$ are continuous. In such a case we can also write $F(x,y) = (f(x), g_x(y))$ where $g_x : I \mapsto I$ is a family of continuous maps depending continuously on x.

Since the triangular map F splits the square I^2 into one-dimensional fibres (intervals $x = \text{const}$) such that each fibre is mapped by F into a fibre, one may expect that the dynamical system (F, I^2) is close, in its dynamical properties, to one-dimensional dynamical systems. In some aspects it is true, e.g., the continuous triangular maps of the square are known to obey the Sharkovsky cycle coexistence ordering [K]. Nevertheless, they prove to have some essential differences if compared with continuous one-dimensional maps (see [KoSh], [Ko]).

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The aim of the present paper is to study ω-limit sets of continuous triangular maps of the unit square into itself. Our main result is the characterization of those ω-limit sets which lie in one fibre (see Theorem 1). The intersection of an ω-limit set with a fibre can be an arbitrary compact subset of the fibre (see Theorem 2).

2. Statement of Main Results

Denote by $C_\Delta(I^2, I^2)$ the set of all continuous triangular maps from I^2 into itself and by $\omega_F([x,y])$ the ω-limit set of the point $[x,y]$ under F. In the present paper we try to find at least partial answer to the question what subsets of the square I^2 can be ω-limit sets for some map from $C_\Delta(I^2, I^2)$.

It is natural to start with the case when a whole ω-limit set is a subset of one fibre. Trivially, as an ω-limit set lying in a fibre $I_a = \{a\} \times I$ we can get any set of the form $\{a\} \times M$ where M is a set which can serve as an ω-limit set for a continuous map $I \mapsto I$. But it turns out that also many other sets can be obtained. The complete answer is given by

Theorem 1 *For $a \in I$, $M \subset I$ the following two conditions are equivalent:*

(i) There is $F \in C_\Delta(I^2, I^2)$ and a point $[x,y] \in I^2$ with $\omega_F([x,y]) = \{a\} \times M$;

(ii) M is a non-empty closed subset of I which is not of the form

$$M = J_1 \cup J_2 \cup \cdots \cup J_n \cup C \tag{1}$$

where n is a positive integer, J_i, $i = 1, 2, \ldots, n$, are closed nondegenerate intervals, C is a non-empty countable set, all the sets J_i and C are mutually disjoint and dist$(C, J_i) > 0$ for at least one $i \in \{1, 2, \ldots, n\}$.

From Theorem 1 and its proof it follows that a non-empty compact subset M of a straight line in the plane is an ω-limit set for a continuous map from the plane into itself if and only if M (considered as an one-dimensional set) is not of the form (1).

Using Theorem 1 it is easy to show that if A is a non-empty finite set then $A \times M$ is an ω-limit set for a continuous triangular map if and only if M is a non-empty closed subset of I which is not of the form (1).

The next step is not to require that an ω-limit set is a subset of a fibre. Then the question is whether any closed subset of a fibre can be obtained as an intersection of this fibre and an ω-limit set of F. The answer is affirmative.

Theorem 2 *Let $a \in I$, and let M be any closed subset of I. Then there are $F \in C_\Delta(I^2, I^2)$ and $[x,y] \in I^2$ with $\omega_F([x,y]) \cap I_a = \{a\} \times M$.*

3. Definitions and Notations

I, I_a and $C_\Delta(I^2, I^2)$ have been defined above. Let $C(X,Y)$ be the set of all continuous maps from X into Y. For every $[x,y] \in I^2$ put $\pi([x,y]) = x$. For a set $\mathcal{K} \subset I^2$ let $C_\Delta(\mathcal{K}, I^2)$ be the set of all continuous triangular maps from $\mathcal{K}$ into I^2. So $F \in C_\Delta(\mathcal{K}, I^2)$ if $F \in C(\mathcal{K}, I^2)$ and $\pi(F(a)) = \pi(F(b))$ whenever $a, b \in \mathcal{K}$ with $\pi(a) = \pi(b)$.

For a compact metric space X and $f \in C(X,X)$ let $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for each $x \in X$ and natural number n. An ω-limit set $\omega_f(x)$ is defined to be the set of limit points of the sequence $\{f^n(x)\}_{n=0}^\infty$. The range of this sequence will be denoted by $\mathrm{orb}_f(x)$. If $\mathcal{A} \subset X$ and $f(\mathcal{A}) \subset \mathcal{A}$ or $f(\mathcal{A}) = \mathcal{A}$, $\mathcal{A}$ is called f-invariant or strongly f-invariant, respectively. Recall that $\omega_f(x)$ is a compact and strongly f-invariant set.

Let $g \in C(I,I)$. A set $\{\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_r\}$ of mutually disjoint subintervals of I is called a g-cycle of intervals if $g(\mathcal{K}_i) = \mathcal{K}_{i+1} \pmod r$. In such a case we write $\mathcal{K}_1 \mapsto \mathcal{K}_2 \mapsto \cdots \mapsto \mathcal{K}_r \mapsto \mathcal{K}_1$ if no confusion can arise by suppressing g.

For $\mathcal{A}, \mathcal{B} \subset I$ let $\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \inf\{|a-b|,\ a \in \mathcal{A},\ b \in \mathcal{B}\}$. Recall that $\mathrm{dist}(\emptyset, \mathcal{A}) = \inf \emptyset = +\infty > 0$. Further, let $\mathrm{clos}\,\mathcal{A}$ be the closure of $\mathcal{A}$, and if $\mathcal{A}$ is closed let $\max \mathcal{A}$ or $\min \mathcal{A}$ be the maximum or minimum of $\mathcal{A}$, respectively. $\mathcal{A} < \mathcal{B}$ means that $a < b$ whenever $a \in \mathcal{A}$, $b \in \mathcal{B}$. If g is a function, then $g|\mathcal{A}$ is the restriction of g to the set $\mathcal{A}$. A set is countable if it is finite or infinite countable.

Sometimes no distinction is made between a point x and a set $\{x\}$. If x is a point then by the midpoint of $\{x\}$ we mean x and in the same way we define the end-points of $\{x\}$. Further, $x \cup \mathcal{A}$ stands for $\{x\} \cup \mathcal{A}$. By $f(M) = m$ where M is a set and m is a point we mean that $f(x) = m$ for all $x \in M$.

Let $\mathcal{F}$ be a system of maps. Denote the domain of f by $\mathcal{D}(f)$ and suppose that $f_1(x) = f_2(x)$ whenever $f_1, f_2 \in \mathcal{F}$ and $x \in \mathcal{D}(f_1) \cap \mathcal{D}(f_2)$. Then one can define a map g with the domain $\cup\{\mathcal{D}(f),\ f \in \mathcal{F}\}$ such that $g|\mathcal{D}(f) = f$ for each $f \in \mathcal{F}$. This map g will be denoted by $\cup \mathcal{F}$. Sometimes we do not state the domains of maps explicitly. Note that if a map f is defined on each of the sets $\mathcal{D}_t$, $t \in \mathcal{T}$ and if it is not stated otherwise, then the domain of f is the set $\bigcup_{t \in \mathcal{T}} \mathcal{D}_t$ and not a larger set. The identity map on a set $\mathcal{A}$ will be denoted by $id_\mathcal{A}$ or, shortly, by id if no confusion can arise by suppressing $\mathcal{A}$.

If $M \subset I$ and $\varepsilon > 0$ then a finite set $\{x_1, x_2, \dots, x_n\} \subset M$ is called an ε-net for M provided that for every $m \in M$ there is x_i with $\mathrm{dist}(m, x_i) < \varepsilon$.

Let $f \in C(M,M)$ and $\varepsilon > 0$. A finite sequence $x_1, x_2, \dots, x_n$ of points from M is said to be an ε-recurrent chain of f or, shortly, an ε-chain of f if, modulo n, $\mathrm{dist}(f(x_i), x_{i+1}) < \varepsilon$ for every $i = 1, 2, \dots, n$.

A non-empty nowhere dense perfect set will be called a Cantor-like set. Recall that by Alexandrov-Hausdorff theorem, any uncountable Borel set contains a

Cantor-like set.

If $M \subset I$ is a nowhere dense compact set then every open interval disjoint with M and having both end-points from M will be said to be an interval contiguous to M.

Finally, note that we use the notation $[x, y]$ to denote both a point in the plane and a closed interval on the real line.

4. Auxiliary Results

Lemma 1 *(Extension lemma) Let $\mathcal{K} \subset I^2$ be a compact set, $\varphi \in C_\Delta(\mathcal{K}, I^2)$. Then there is a map $\Phi \in C_\Delta(I^2, I^2)$ such that for every $[x,y] \in \mathcal{K}$ $\Phi(x,y) = \varphi(x,y)$.*

PROOF. Denote $\varphi(x,y) = (f(x), g(x,y))$. Since $\varphi \in C(\mathcal{K}, I^2)$, we have $g \in C(\mathcal{K}, I)$. We are going to prove that also $f : \pi(\mathcal{K}) \mapsto I$ is continuous. (This is not true without the assumption that the set $\mathcal{K}$ is compact.) Assume, on the contrary, that f is discontinuous at a point $x \in \pi(\mathcal{K})$. Then there is a sequence of points $x_n \in \pi(\mathcal{K})$ for which $x_n \to x$ and $f(x_n) \not\to f(x)$. Since I is a compact interval, there is a converging subsequence of $f(x_n)$. Without loss of generality we may assume that $f(x_n) \to a \neq f(x)$. Take points y_n with $[x_n, y_n] \in \mathcal{K}$. There is a converging subsequence of y_n with $[x_n, y_n] \in \mathcal{K}$. There is a converging subsequence of y_n. Without loss of generality we may assume that $y_n \to y$. Then $[x_n, y_n] \to [x,y]$. Since $\mathcal{K}$ is closed, $[x,y] \in \mathcal{K}$. The point $\varphi(x,y)$ belongs to the fibre $I_{f(x)}$ and the sequence $f(x_n)$ does not converge to $f(x)$. So $\varphi(x_n, y_n)$ does not converge to $\varphi(x,y)$, and we have a contradiction with the continuity of φ.

By Tietze extension theorem the functions $f \in C(\pi(\mathcal{K}), I)$ and $g \in C(\mathcal{K}, I)$ have continuous extensions $F \in C(I, I)$ and $G \in C(I^2, I)$, respectively. Now it suffices to put $\Phi(x,y) = (F(x), G(x,y))$. $\square$

Lemma 2 *Let $a \in I$, $M \subset I$ be a closed set, $h \in C(M, M)$. Suppose that for every $\varepsilon > 0$ there is an ε-chain of h which is an ε-net for M. Then there are $F \in C_\Delta(I^2, I^2)$ and $[x,y] \in I^2$ with $\omega_F([x,y]) = \{a\} \times M$.*

PROOF. Without loss of generality we may assume that $a < \max I$. Denote $m = \min M$. It follows from the assumptions that for every $\varepsilon > 0$ there is an ε-chain of h which is an ε-net for M and contains the point m. Without loss of generality we may assume that these chains start at the point m. Take a sequence ε_n, $n = 1, 2, \ldots$, $\varepsilon_n \searrow 0$ and a sequence c_n, $n = 1, 2, \ldots$, where c_n is an ε_n-chain of h which is an ε_n-net for M and starts at m. Denote $c_1 = \{m = y_1, y_2, \ldots, y_{k(1)}\}$, $c_2 = \{m = y_{k(1)+1}, \ldots, y_{k(1)+k(2)}, \ldots, c_n = \{m = y_{k(1)+\cdots+k(n-1)+1}, \ldots, y_{k(1)+\cdots+k(n)}\}, \ldots$. Take a sequence x_n, $n =$

$1, 2, \ldots,\quad x_n \searrow a$ and the sequence $A_n = [x_n, y_n]$, $n = 1, 2, \ldots$. Denote the set $(\{a\} \times M) \cup \{A_n,\ n = 1, 2, \ldots\}$ by $\mathcal{K}$ and define a function φ from $\mathcal{K}$ into itself as follows: $\varphi(A_n) = A_{n+1}$, $n = 1, 2, \ldots$, and $\varphi([a, y]) = [a, h(y)]$ whenever $y \in M$. Then $\mathcal{K} = (\{a\} \times M) \cup \mathrm{orb}_\varphi([x_1, y_1])$.

It is not hard to see that $\omega_\varphi([x_1, y_1] = \{a\} \times M$. The inclusion $\omega_\varphi([x_1, y_1]) \subset \{a\} \times M$ follows from the facts that $x_n \rightarrow a$ and for every n, $y_n \in M$. To prove the converse inclusion it suffices to take into consideration that for every n, $c_n \subset M$ is an ε_n-net for M. So $\mathcal{K} = \omega_\varphi([x_1, y_1]) \cup \mathrm{orb}_\varphi([x_1, y_1])$ is a compact set. The function φ is triangular. Further, φ is continuous at each point A_n and since h is continuous, $x_n \rightarrow a$ and for every n, c_n is an ε_n-chain of h with $\varepsilon_n \rightarrow 0$, φ is also continuous at each point from $\{a\} \times M$. Now by Lemma 1 we get a function $F \in C_\Delta(I^2, I^2)$ with $\omega_F([x_1, y_1]) = \{a\} \times M$. □

In the sequel we will write $h \in M(\mathcal{E})$ whenever $h \in C(M, M)$ is such that for every $\varepsilon > 0$ there is an ε-chain of h which is an ε-net for M. Further we will write $M \in \mathcal{E}$ whenever there is an $h \in M(\mathcal{E})$. So Lemma 2 says that if $M \in \mathcal{E}$ is a closed set then $\{a\} \times M$ is an ω-limit set for a triangular map.

Lemma 3 *Let (X, ρ) be a compact metric space, $f \in C(X, X)$, $M \subset X$, $M = M_1 \cup M_2$, $M_1, M_2 \neq \emptyset$, $\rho(M_1, M_2) > 0$. If $f(M_1) \subset M_1$ then there is no point $x_0 \in X$ with $\omega_f(x_0) = M$.*

PROOF. This is an easy consequence of Theorem 1 from [Sh] saying that if (X, ρ) is a compact metric space, $f \in C(X, X)$, $x_0 \in X$, $\mathcal{U}$ is an open subset of $\omega_f(x_0)$ (in relative topology), and $\mathcal{U} \neq \omega_f(x)$ then the closure of $f(\mathcal{U})$ is not contained in $\mathcal{U}$. □

In the sequel, for any two subsets $\mathcal{A}$, $\mathcal{B}$ of I, $\mathcal{A} \succ \mathcal{B}$ means that there is a continuous map of $\mathcal{A}$ onto $\mathcal{B}$. In [BS] it is proved that if $\mathcal{A}$, $\mathcal{B} \subset I$ are nowhere dense compact sets, $\mathcal{A}$ uncountable and $\mathcal{B} \neq \emptyset$, then $\mathcal{A} \succ \mathcal{B}$. We shall need the following stronger result.

Lemma 4 *Let $\mathcal{B} \subset I$ be a non-empty compact set and $\mathcal{A} \subset I$ be a compact set containing a Cantor-like set P such that no interval contiguous to P is a subset of $\mathcal{A}$. Then $\mathcal{A} \succ \mathcal{B}$.*

PROOF. Since $P \succ I$, there is a compact subset $Q \subset P$ with $Q \succ \mathcal{B}$. It suffices to show that $\mathcal{A} \succ Q$. Every interval $J = (q', q'')$ contiguous to Q contains an interval contiguous to P and consequently a point which does not belong to the closed set $\mathcal{A}$. Hence there are disjoint compact intervals $J' = (q', a')$ and $J'' = (a'', q'')$ such that $A \cap J \subset J' \cup J''$. Some of these two intervals may intersect the set $\mathcal{A} \setminus Q$. Using this it is easy to see that there exists a countable system of compact intervals J_n such that for any $m \neq n$, $J_n \cap Q = \{q_n\}$, $J_m \cap J_n \subset Q$, $J_n \cap (\mathcal{A} \setminus Q) \neq \emptyset$ and $\mathcal{A} \setminus Q \subset \bigcup_n J_n$. Now let φ be the identity map on Q, and let φ be constant on every $J_n \cap \mathcal{A}$. Clearly φ is continuous and $\varphi(\mathcal{A}) = Q$. $\square$

In [BS] it is defined what it means for a nowhere dense compact set M to be homoclinic with respect to a continuous map. We will not assume that M is nowhere dense. So let $M \subset I$ be a compact set, and let $\mathcal{A} = \{a_0, \dots, a_{k-1}\} \neq \emptyset$ be a set of points of M. Assume that there is a system $\{I_n^i\}_{n=0}^{\infty}$, $i = 0, \dots, k-1$ of pairwise disjoint compact intervals such that $M \setminus \bigcup_{i,n} I_n^i = \mathcal{A}$, $M_n^i = M \cap I_n^i \neq \emptyset$ for every i, n, and $\lim_{n\to\infty} M_n^i = a_i$ for any i (i.e., every neighborhood of a_i contains the sets M_n^i for all sufficiently large n). Let $f \in C(M, M)$, and let $\mathcal{A}$ be a k-cycle of f such that $f(a_i) = a_{i-1}$ for $i > 0$ and $f(a_0) = a_{k-1}$. If $f(M_n^i) = M_n^{i-1}$ for $i > 0$ and any n, $f(M_n^0) = M_{n-1}^{k-1}$ for $n > 0$, and $f(M_0^0) = a_{k-1}$, then M is called a homoclinic set (of order k) with respect to f. In the sequel, the sets M_n^i or the cycle A will be called the portions of M or the initial cycle of M, respectively. If $\mathcal{A} = \{a\}$, then a will be called the initial point of M. The portion M_0 with $f(M_0) = a$ will be called the last portion of M.

Clearly, if M is homoclinic with respect to f then $f \in M(\mathcal{E})$ and thus $M \in \mathcal{E}$.

Lemma 5 *(See the proof of Lemma 4 from [BS]). Let M be an uncountable nowhere dense compact subset of I, and let either a be a bilateral condensation point of M or $a \in \{\min M, \max M\}$ be a condensation point of M. Then there is a continuous map f from M onto itself such that M is homoclinic with respect to f, a is the initial point of M, $f(a) = a$, and all the portions M_n, $n = 0, 1, \dots$, are uncountable. Consequently, $M \in \mathcal{E}$.*

Lemma 6 *Let $M \subset I$ be a compact set containing a Cantor-like set P such that no interval contiguous to P is a subset of M. Let a be a bilateral condensation point of P. Then there is a continuous map f from M onto itself such that M is homoclinic with respect to f, a is the initial point of M, $f(a) = a$, and every portion M_n, $n = 0, 1, 2, \dots$ is a compact set containing a Cantor-like set P_n such that no interval contiguous to P_n is a subset of M_n. Consequently, $M \in \mathcal{E}$.*

PROOF. Let J_n, $n = 0, 1, 2, \dots$ be disjoint compact intervals such that $\bigcup_{n=0}^{\infty} J_n \supset P \setminus \{a\}$, $\lim_{n\to\infty} J_n = a$, $J_n \cap P = P_n$ are Cantor-like sets and $\{\min J_n, \max J_n\} \subset P$. Since no interval contiguous to P is a subset of M, there are disjoint compact intervals $\mathcal{K}_n$ with $\mathcal{K}_n \supset J_n$ and $\bigcup_{n=0}^{\infty} \mathcal{K}_n \supset M \setminus \{a\}$. Then for every n, $M_n = \mathcal{K}_n \cap M$ is a compact set containing a Cantor-like set P_n such that no interval contiguous to P_n is a subset of M_n. By Lemma 4, for every n there is a continuous map f_n from M_{n+1} onto M_n. To finish the proof take $f = \bigcup_{n=0}^{\infty} f_n$ and extend f to the set M by putting $f(a) = a$ and $f(M_0) = a$. $\square$

Lemma 7 *Let $M \subset I$ be a compact set having uncountably many connected components. Then $M \in \mathcal{E}$.*

PROOF. Only countably many of the components of M are intervals. Denote their union by $\mathcal{A}$. Then $M = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{B}$ is disjoint with $\mathcal{A}$ and uncountable. Take a Cantor-like set $P \subset \mathcal{B}$. Then no interval contiguous to P is a subset of M and by Lemma 6, $M \in \mathcal{E}$. $\square$

Before stating next lemma we need some notation. Let $\mathcal{A} \subset I$ be a countable compact set. Define a transfinite sequence $\{\mathcal{A}_\alpha\}_{\alpha<\Omega}$ of subsets of $\mathcal{A}$ as follows: $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_\gamma = \bigcap_{\alpha<\gamma} \mathcal{A}_\alpha$ if γ is a limit ordinal, and A_γ is the derivative (i.e., the set of limit points) of $\mathcal{A}_{\gamma-1}$ otherwise. Clearly, for any such $\mathcal{A}$ there is an ordinal $\beta < \Omega$ such that $\mathcal{A}_\beta$ is non-empty and finite (and hence, $\mathcal{A}_{\beta+1} = \emptyset$). This β is called the depth of $\mathcal{A}$ and is denoted by $d(\mathcal{A})$. The set $\mathcal{A}_\beta$ is said to be the kernel of $\mathcal{A}$. Instead of $\mathcal{A}_\beta$ we also use the symbol $\mathrm{Ker}(\mathcal{A})$. The points from $\mathcal{A}_\alpha \setminus \mathcal{A}_{\alpha+1}$ are said to have depth α with respect to $\mathcal{A}$. The depth of a point x with respect to $\mathcal{A}$ is denoted by $d(x|\mathcal{A})$. Clearly, if a point $x \in \mathcal{A}$ has depth α (with respect to $\mathcal{A}$, then there is a punctured neighborhood $\mathcal{U}$ of x (i.e., a neighborhood of x without the point x) such that all points from $\mathcal{U} \cap \mathcal{A}$ have depths less than α (with respect to $\mathcal{A}$). Otherwise x would have depth at least $\alpha + 1$. So there is a neighborhood $V = U \cup \{x\}$ of x such that $\mathrm{Ker}(V \cap \mathcal{A}) = \{x\}$.

Lemma 8 *(See Lemma 6 and its proof in [BS].) Let $\mathcal{A}, \mathcal{B}$ be countable compact sets with $d(\mathcal{A}) \geq d(\mathcal{B})$, and let $Ker\,\mathcal{B} = \{b\}$. Then there is a continuous map f from $\mathcal{A}$ onto $\mathcal{B}$ such that $f(Ker\,\mathcal{A}) = Ker\,\mathcal{B}$.*

Lemma 9 *(See the proof of Theorem 3 in [BS].) Let $\mathcal{A}$ be an infinite countable compact subset of I. Then there is a continuous map f from $\mathcal{A}$ onto itself such that $\mathcal{A}$ is homoclinic with respect to f and $Ker\,\mathcal{A}$ is the initial cycle of $\mathcal{A}$.*

Lemma 10 *Let $\mathcal{K} \subset I$ be a compact set of the form $\mathcal{K} = J \cup C$ where J is a compact interval or a singleton, C is non-empty countable and disjoint with J, and $dist(C, J) = 0$. Then there exist a non-empty compact set $L \subset C$ with $dist(L, \mathcal{K} \setminus L) > 0$ and a map $g \in \mathcal{K}(\mathcal{E})$ such that $g|J$ is the identity map and $g(L)$ is the midpoint of J. Consequently, $\mathcal{K} \in \mathcal{E}$. (In what follows, the set L will be called the last portion of $\mathcal{K}$ with respect to g.)*

PROOF. Denote by m the midpoint of J and by C^+ or C^- the set of all $x \in C$ with $x > \max J$ or $x < \min J$, respectively. We distinguish two cases.

Case 1. Only one of the sets C^+ and C^- has zero distance from J. Without loss of generality we may assume that $\mathrm{dist}(C^+, J) = 0$ and C^- is either empty or non-empty and $\mathrm{dist}(C^-, J) > 0$.

Since the point $\max J$ is limit for C^+, the set $\mathrm{clos}\, C = C \cup \max J$ can be expressed in the form $\mathrm{clos}\, C = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are disjoint, $C^- \subset \mathcal{B}$, $\mathcal{A} \cap \mathrm{clos}\, C^+ < \mathcal{B} \cap \mathrm{clos}\, C^+$, A is an infinite countable compact set, $\mathrm{Ker}\,\mathcal{A} = \{\max J\}$, and $\mathcal{B}$ is a countable compact set. Even in the case when C^- is empty we may, without loss of generality, assume that $\mathcal{B}$ is non-empty (finite or infinite countable).

By Lemma 9 there is a continuous map $f_{\mathcal{A}}$ from $\mathcal{A}$ into itself such that $\mathcal{A}$ is homoclinic with respect to $f_{\mathcal{A}}$ with the initial point $\max J$ and a last portion $\mathcal{A}_0$. Denote $g = f_{\mathcal{A}}|\mathcal{A} \setminus \mathcal{A}_0$. We are going to extend g to the set $\mathcal{K}$. First of all, define $g(x) = x$ for every $x \in J$.

If $\mathcal{B} = \{b_1, b_2, \ldots, b_r\}$ is finite, put $g(\mathcal{A}_0) = b_1$, $g(b_i) = b_{i+1}$, $i = 1, 2, \ldots, r-1$, and $g(b_r) = m$. Here $L = \{b_r\}$.

If $\mathcal{B}$ is infinite, then we can use Lemma 9 to obtain a continuous map $f_{\mathcal{B}}$ from $\mathcal{B}$ onto itself such that $\mathcal{B}$ is homoclinic with respect to $f_{\mathcal{B}}$ with an initial cycle P and a last portion L. Take a point $p \in P$. Put $g(\mathcal{A}_0) = p$, $g(x) = f_{\mathcal{B}}(x)$ for every $x \in \mathcal{B} \setminus L$ and $g(L) = m$.

In every case we have found a last portion L of $\mathcal{K}$ and a continuous map g from $\mathcal{K}$ onto itself with the desired properties. Since $g \in \mathcal{K}(\mathcal{E})$ we have $\mathcal{K} \in \mathcal{E}$.

Case 2. Both the sets C^+ and C^- have zero distance from J.

Denote $Q^+ = J \cup C^+$, $Q^- = J \cup C^-$. From Case 1 which has already been proved we know that there are continuous maps g^+ and g^- with $g^+ \in Q^+(\mathcal{E})$ and $g^- \in Q^-(\mathcal{E})$. Since $g^-|J = g^+|J$ we can define $g = g^+ \cup g^-$. Then $g \in \mathcal{K}(\mathcal{E})$, and thus $\mathcal{K} \in \mathcal{E}$. In the considered Case 2 we define the last portion of $\mathcal{K}$ with respect to g to be that of Q^+ with respect to g^+. □

Now let M be a compact subset of I of the form

$$M = \bigcup_{n=1}^{\infty} J_n \cup C \tag{2}$$

where all the sets C and J_i, $i = 1, 2, \ldots$, are mutually disjoint, J_i, $i = 1, 2, \ldots$, are compact intervals, and C is a countable set (empty or non-empty). Clearly, C is nowhere dense. Denote $J = \text{clos}(\bigcup_{n=1}^{\infty} J_n)$. Then J is compact, and since $\bigcup_{n=1}^{\infty} J_n \subset J \subset M$, both the sets $M \setminus J$ and $J \setminus \bigcup_{n=1}^{\infty} J_n$ are countable.

Consider the map h from I into itself defined by $h(x) = x - \lambda([0,x] \cap \bigcup_{n=1}^{\infty} J_n)$, where λ is the Lebesgue measure. Any component of J is either a point from $J \setminus \bigcup_{n=1}^{\infty} J_n$ or an interval J_n for some n. A component x of J is said to be limit provided that $h(x)$ is a limit point of the set $h(J)$. Similarly, we define a limit component from the right or left.

Clearly, J has at least one limit component. The depth of a component x of J with respect to J is defined to be that of the point $h(x)$ with respect to $h(J)$ and is denoted by $d(x|J)$. A component x of J having zero depth is necessarily an interval J_n and has a positive distance from $J \setminus J_n$. Finally, define the depth of J, $d(J) = d(h(J))$ and the kernel of J, $\text{Ker}\, J = h^{-1}(\text{Ker}\, h(J))$.

Lemma 11 *Let $M \imath I$ be a compact set of the form (2) such that the set $J = clos\,(\bigcup_{n=1}^{\infty} J_n)$ has only one limit component P. Then there exists a non-empty* *compact set $L \imath M$ with dist$(L, M\backslash L) > 0$ and a map $g \in M(\mathcal{E})$ such that $g|P$ is the identity map and $g(L)$ is the midpoint of P. Consequently, $M \in \mathcal{E}$. (In what follows, the set L will be called the last portion of M with respect to g.)*

PROOF. *(a) Reduction of the problem.* First of all we are going to show that we can, without loss of generality, assume that $P < M\backslash P$.

Suppose we have proved the lemma when the component P is limit only from one side. Then the lemma is also true if P is limit from both sides. In fact, one can take $M^+ = \{x \in M : x \geq \min P\}$, $M^- = \{x \in M : x \leq \max P\}$, the corresponding maps $g^+ \in M^+(\mathcal{E})$ and $g^- \in M^-(\mathcal{E})$ and define $g = g^+ \cup g^-$. Finally, one can take the last portion of M^+ with respect to g^+ as the last portion of M with respect to g.

So let P be limit only from one side, say, from the right. Suppose we have proved the lemma when dist$(C^-, P) > 0$, where $C^- = \{x \in C : x < \min P\}$. Then the lemma is also true if dist$(C^-, P) = 0$. In fact, in this case take an interval $[\min P - \delta, \ \min P[$ meeting no J_n and having a positive distance from the set $\{x \in M : x < \min P - \delta\}$. Denote $\mathcal{K} = \{\min P\} \cup C_0^-$, where $C_0^- = \{x \in C : \min P - \delta \leq x < \min P\}$. By Lemma 10, there is a map $g_1 \in \mathcal{K}(\mathcal{E})$ leaving $\min P$ fixed. Denote $Q = M\backslash C_0^-$. According to our assumption, the lemma holds if we take Q instead of M. So there is a map $g_2 \in Q(\mathcal{E})$ such that $g_2|P$ is the identity and a set L is the last portion of Q with respect to g_2. Then $g = g_1 \cup g_2$ belongs to $M(\mathcal{E})$, $g|P$ is the identity, and L can be taken as the last portion of M.

Thus, we have shown that we can restrict ourselves to the case when P is limit only from the right and dist$(C^-, P) > 0$, i.e. M is of the form $M = M_1 \cup P \cup M_2$, $M_1 < P < M_2$ and dist$(M_1, P) > 0$. But, obviously, the lemma holds for the sets of such a form if and only if it holds for the sets of the form $M = P \cup M_2 \cup M_1$, $P < M_2 < M_1$ and dist$(M_2, M_1) > 0$.

We have reduced our problem to the following one: Prove the lemma under the additional assumption that $P < M\backslash P$.

(b) Proof of the reduced problem. Let, additionally, $P < M\backslash P$. The system of those intervals J_n, $n = 1, 2, \ldots$, which are different from P can be divided into two systems A and B as follows: If dist(J_n, C) is positive or zero, then $J_n \in A$ or $J_n \in B$, respectively. Let $B^{\pm}$ be the system of those intervals from B whose both end-points are limit for the set C. If only the right or left endpoint of an interval from B is limit for C, then let it belong to B^+ or B^-, respectively. If the right end-point of an interval $\mathcal{B} \in B$ is limit for C, then there is a neighborhood of $\max \mathcal{B}$ intersecting C in a set $C^+(\mathcal{B})$ such that $C^+(\mathcal{B})$ has a positive distance from $M\backslash(\mathcal{B} \cup C^+(\mathcal{B}))$ and all the points from $C^+(\mathcal{B})$ have their depths with respect to clos $C^+(\mathcal{B}) = \{\max \mathcal{B}\} \cap C^+(\mathcal{B})$ less than the point $\max \mathcal{B}$ has. If $\max \mathcal{B}$ is not a limit point for C, put $C^+(\mathcal{B}) = \emptyset$. The set $C^-(\mathcal{B})$ is defined analogously, and $C(\mathcal{B}) = C^+(\mathcal{B}) \cap C^-(\mathcal{B})$.

Now suppose that the system B is infinite. Then at least one of the systems B^+, B^- and $B^\pm$ is infinite. We can assume that $B^\pm$ is infinite. (If not, we proceed analogously with B^+ or B^- instead of $B^\pm$. Then the procedure is even less complicated than now. In the sequel, we will always assume that $B^\pm$ is infinite whenever B is infinite.) Consider the set

$$\mathcal{S} = \{\max P\} \cup \bigcup_{\mathcal{B}\in B^\pm} (\{\max \mathcal{B}\} \cup C^+(\mathcal{B}))$$

and denote $d(\max P|\mathcal{S}) = m$. All points from $\mathcal{S}$ lying in a punctured neighborhood of max P have their depths with respect to $\mathcal{S}$ less than m. Further, if $m' < m$, then in any punctured neighborhood of max P there is a $\mathcal{B} \in B^\pm$ such that $d(\max \mathcal{B}|\mathcal{S}) \geq m'$ (in the opposite case it would be $d(\max P|\mathcal{S}) \leq m'$.) It follows from this that there is a sequence of intervals from $B^\pm$ converging to max P such that the depths of maxima of these intervals with respect to $\mathcal{S}$ form a non-decreasing sequence. Further, realize that $d(\max \mathcal{B}|\mathcal{S}) = d(\max \mathcal{B}|\{\max \mathcal{B}\} \cup C^+(\mathcal{B}))$. Similarly, as we have chosen the sequence of intervals from $B^\pm$, we can choose a subsequence from this sequence such that the depths of minima of the intervals from the subsequence form a non-decreasing sequence, too. (Here the depth of min $\mathcal{B}$ is taken with respect to $\{\min \mathcal{B}\} \cup C^-(\mathcal{B})$.)

As a result of this consideration we can see that if B is infinite, say, if $B^\pm$ is infinite (the two other cases are similar to this one), then we can write $B = B_1 \cup B_2$ where $B_2 = B \setminus B_1$, $B_1 = \{\mathcal{B}_n^1,\ n = 1, 2, \ldots\}$, $\lim_{n\to\infty} \mathcal{B}_n^1 = \max P$ and the sequence $\{d(\min \mathcal{B}_n^1|\{\min \mathcal{B}_n^1\} \cup C^-(\mathcal{B}_n^1))\}_{n=1}^\infty$ as well as the analogous sequence for maxima, is non-decreasing. Here we can, without loss of generality, assume that the system B_2 is infinite. Finally, recall that for every n, $\{\min \mathcal{B}_n^1\}$ is the kernel of clos $C^-(\mathcal{B}_n^1) = \{\min \mathcal{B}_n^1\} \cup C^-(\mathcal{B}_n^1)$ (and similarly for maxima).

All things considered, we need to prove the lemma when $P < M \setminus P$ and

$$M = P \cup \bigcup_{\mathcal{A}\in A} \mathcal{A} \cup \bigcup_{\mathcal{B}\in B} (\mathcal{B} \cup C(\mathcal{B})) \cup \mathcal{D} \tag{3}$$

where $\mathcal{D} = C \setminus (P \cup \bigcup_{\mathcal{B}\in B} C(\mathcal{B}))$. If B is infinite then $B = B_1 \cup B_2$.

Now we are going to define seven maps which will be useful later.

(o) (Definition of φ_0, s_0 and L_0 when A is finite and non-empty and B is infinite.) Let $A = \{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$, and let B be infinite. Consider the system $B_1 = \{\mathcal{B}_n^1,\ n = 1, 2, \ldots\}$ described above. By Lemma 8, there are continuous maps g_n and h_n, $n = 1, 2, \ldots$, such that $g_1(\text{clos } C^+(\mathcal{B}_1^1)) = \max \mathcal{A}_1$, $h_1(\text{clos } C^-(\mathcal{B}_1^1)) = \min \mathcal{A}_1$ and for $n = 2, 3, \ldots$, $g_n(\text{clos } C^+(\mathcal{B}_n^1)) = \text{clos } C^+(\mathcal{B}_{n-1}^1)$, $g_n(\max \mathcal{B}_n^1) = \max \mathcal{B}_{n-1}^1$, $h_n(\text{clos } C^-(\mathcal{B}_n^1)) = \text{clos } C^-(\mathcal{B}_{n-1}^1)$, $h_n(\min \mathcal{B}_n^1) = \min \mathcal{B}_{n-1}^1$. Further, let f be the map such that $f|P = id$, f is linear and increasing on each

of the intervals $\mathcal{B}_n^1$, $n = 1, 2, \dots$ and $\mathcal{A}_i$, $i = 1, 2, \dots, r-1$ and $f(\mathcal{B}_n^1) = \mathcal{B}_{n-1}^1$ for $n = 2, 3, \dots$, $f(\mathcal{B}_1^1) = \mathcal{A}_1$ and $f(\mathcal{A}_i) = \mathcal{A}_{i+1}$, $i = 1, 2, \dots, r-1$. Now let $\varphi_0 = f \cup \bigcup_{n=1}^{\infty}(g_n \cup h_n)$. Since P is the only limit component of J, $\lim_{n\to\infty} B_n^1 = \max P$. Thus φ_0 is a continuous map from $P \cup \bigcup_{\mathcal{B}\in B_1}(\mathcal{B}\cup C(\mathcal{B}))\cup(\mathcal{A}_1\cup \dots \cup\mathcal{A}_{r-1})$ onto $P \cup \bigcup_{\mathcal{B}\in B_1}(\mathcal{B}\cup C(\mathcal{B}))\cup \bigcup A$. Finally, denote $\mathcal{A}_r$ by L_0 and the midpoint of P by s_0.

(i) (Definition of φ_1 and L_1 when A is finite and non-empty.) Let $A = \{\mathcal{A}_1, \dots, \mathcal{A}_r\}$, and let φ be a continuous map from $(\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{r-1})$ onto $(\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{r-1})$ such that $\varphi_1|\mathcal{A}_i$ is linear and $\varphi_1(\mathcal{A}_i) = \mathcal{A}_{i+1}$, $i = 1, 2, \dots, r-1$. Denote $\mathcal{A}_r$ by L_1.

(ii) (Definition of φ_2, s_2 and L_2 when A is infinite.) Let $A = \{\mathcal{A}_1, \mathcal{A}_2, \dots\}$. Define a map φ_2 from $(P \cup \bigcup A)\setminus\mathcal{A}_1$ onto $P \cup \bigcup A$ such that $\varphi_2|P$ is the identity map, $\varphi_2|\mathcal{A}_i$ is linear and $\varphi_2(\mathcal{A}_i) = \mathcal{A}_{i-1}$, $i = 2, 3, \dots$. Clearly, φ_2 is continuous. Denote the midpoint of P by s_2 and $\mathcal{A}_1$ by L_2.

(iii) (Definition of φ_3, s_3 and L_3 when B is finite and non-empty.) Let $B = \{\mathcal{B}_1, \dots, \mathcal{B}_s\}$. Denote $\mathcal{K}_i = \mathcal{B}_i \cup C(\mathcal{B}_i)$, $i = 1, \dots, s$. According to Lemma 10, for every $i = 1, 2, \dots, s$ there is a map $g_i \in \mathcal{K}_i(\mathcal{E})$ and a last portion $\mathcal{H}_i$ of $\mathcal{K}_i$ such that $g_i|\mathcal{B}_i = id$ and $g_i(\mathcal{H}_i) = m_i$, where m_i is the midpoint of $\mathcal{B}_i$. Denote $f_i = g_i|\mathcal{K}_i\setminus\mathcal{H}_i$, $i = 1, \dots, s$, and define h_i by $h_i(\mathcal{H}_i) = m_{i+1}$, $i = 1, \dots, s-1$. Then $\varphi_3 = \bigcup_{i=1}^{s} f_i \cup \bigcup_{i=1}^{s-1} h_i$ is a continuous map from $\bigcup_{i=1}^{s}\mathcal{K}_i\setminus\mathcal{H}_s$ onto $\bigcup_{i=1}^{s}\mathcal{K}_i$. Finally, denote the midpoint of $\mathcal{B}_1$ by s_3 and $\mathcal{H}_s$ by L_3.

(iv) (Definition of φ_4, s_4 and L_4 when B is infinite.) Let $B = \{\mathcal{B}_1, \mathcal{B}_2, \dots\}$, and let $\mathcal{K}_i, m_i, \mathcal{H}_i, g_i$ and f_i be defined as in *(iii)* (now for all $i = 1, 2, \dots$). Further, define q_i by $q_i(\mathcal{H}_i) = m_{i-1}$ for $i = 2, 3, \dots$. Let f_0 be the identity map on P. Then $\varphi_4 = \bigcup_{i=0}^{\infty} f_i \cup \bigcup_{i=2}^{\infty} q_i$ is a continuous map from $(P \cup \bigcup_{i=1}^{\infty}\mathcal{K}_i)\setminus\mathcal{H}_1$ onto $P \cup \bigcup_{i=1}^{\infty}\mathcal{K}_i$. Finally, denote the midpoint of P by s_4 and $\mathcal{H}_1$ by L_4.

(v) (Definition of φ_5, s_5 and L_5 when $\mathcal{D}$ is non-empty and has a positive distance from P.) Let $\mathcal{D} \neq \emptyset$ and $\mathrm{dist}(\mathcal{D}, P) > 0$. If $\mathcal{D}$ is finite, $\mathcal{D} = \{d_1, \dots, d_t\}$, then define φ_5 from $\{d_1, \dots, d_{t-1}\}$ onto $\{d_2, \dots, d_t\}$ by $\varphi_5(d_i) = d_{i+1}$, $i = 1, 2, \dots, t-1$ and denote $s_5 = d_1$, $L_5 = \{d_t\}$. Now let $\mathcal{D}$ be infinite. By Lemma 9, there is a continuous map f from $\mathcal{D}$ onto itself such that $\mathcal{D}$ is homoclinic with respect to f. Take a point from the initial cycle of $\mathcal{D}$ and denote it by s_5. Denote the last portion of $\mathcal{D}$ with respect to f by L_5 and put $\varphi_5 = f|\mathcal{D}\setminus L_5$. Then φ_5 is a continuous map from $\mathcal{D}\setminus L_5$ onto $\mathcal{D}$.

(vi) (Definition of φ_6, s_6 and L_6 when $\mathcal{D}$ has zero distance from P.) Suppose that $\mathrm{dist}(\mathcal{D}, P) = 0$. Then there is a decomposition $\mathcal{D} = \bigcup_{i=1}^{\infty}\mathcal{D}_i$ such that $P < \dots < \mathcal{D}_n < \dots < \mathcal{D}_2 < \mathcal{D}_1$ and $\mathrm{dist}(\mathcal{D}_i, \mathcal{D}_{i+1}) > 0$ for every i. Clearly, $\mathcal{D}_i$, $i = 1, 2, \dots$, are countable compact sets, and $\lim_{n\to\infty}\mathcal{D}_n = \max P$. Let n be any positive integer. By *(v)*, there are a point $s_5^n \in \mathcal{D}_n$, a compact set $L_5^n \imath \mathcal{D}_n$ and a continuous map φ_5^n from $\mathcal{D}_n\setminus L_5^n$ onto $\mathcal{D}_n\setminus\{s_5^n\}$ (if $\mathcal{D}_n$ is finite) or onto $\mathcal{D}_n$ (if $\mathcal{D}_n$ is infinite). Further, for $n = 2, 3, \dots$ define ψ_n by $\psi_n(L_5^n) = s_5^{n-1}$. Let

f_0 be the identity map on P. Then $\varphi_6 = f_0 \bigcup_{n=1}^{\infty} \varphi_5^n \cup \bigcup_{n=2}^{\infty} \psi_n$ is a continuous map from $P \cup \mathcal{D} \setminus L_5^1$ onto $P \cup \mathcal{D}$. Finally, denote the midpoint of P by s_6 and L_5^1 by L_6.

Now we are ready to finish the proof of the lemma. The following notation will be useful: If W is a set and w is a point, then $\langle W \to w \rangle$ denotes the constant map f defined on W such that $f(W) = w$.

To finish the proof, recall that M is of the form (3) and distinguish the following three cases.

Case 1. *A is empty.* Then B is infinite. Define

$$g = \varphi_4 \cup \langle L_4 \to s_i \rangle \cup \varphi_i \cup \langle L_i \to s_4 \rangle$$

where $i = 4$ if $\mathcal{D}$ is empty, $i = 5$ if $\mathcal{D}$ is non-empty and $\operatorname{dist}(\mathcal{D}, P) > 0$, and $i = 6$ if $\operatorname{dist}(\mathcal{D}, P) = 0$. Finally, put $L = L_i$. Then it is easy to see that L and g have all the desired properties.

Case 2. *A is infinite.* Define

$$g = \varphi_2 \cup \langle L_2 \to s_i \rangle \cup \varphi_i \cup \langle L_i \to s_j \rangle \cup \varphi_j \cup \langle L_j \to s_2 \rangle$$

where

$$\begin{array}{lll} i = 2 & \text{if} & B \text{ is empty,} \\ i = 3 & \text{if} & B \text{ is non-empty and finite,} \\ i = 4 & \text{if} & B \text{ is infinite,} \\ j = i & \text{if} & \mathcal{D} \text{ is empty,} \\ j = 5 & \text{if} & \mathcal{D} \text{ is non-empty and } \operatorname{dist}(\mathcal{D}, P) > 0, \text{ and} \\ j = 6 & \text{if} & \operatorname{dist}(\mathcal{D}, P) = 0. \end{array}$$

Put $L = L_j$. Again, L and g have all the desired properties.

Case 3. *A is non-empty and finite.* Then B is infinite. We can write (see(3)) $M = M_0 \cup M_1$ where

$$M_0 = P \cup \bigcup_{\mathcal{B} \in B_1} (\mathcal{B} \cup C(\mathcal{B})) \cup \bigcup_{\mathcal{A} \in A} \mathcal{A}$$

and

$$M_1 = P \cup \bigcup_{\mathcal{B} \in B_2} (\mathcal{B} \cup C(\mathcal{B})) \cup \mathcal{D}$$

The proof will be shorter if we use the fact that, without loss of generality, we may assume that B_2 is infinite. Then M_1 is of the form (2), and P is the only limit component of the set $\operatorname{clos} \bigcup_{\mathcal{B} \in B_2} \mathcal{B}$. So, by Case 1, the lemma holds for M_1, i.e., there exists a map $f \in M_1(\mathcal{E})$ such that $f|P = id$. Further, consider φ_0, s_0 and L_0 from *(o)*. Then

$$g = f \cup \varphi_0 \cup \langle L_0 \rightarrow s_0\rangle$$

and $L = L_0$ have all the desired properties.

The proof of the lemma is complete. □

Lemma 12 *Let $M \imath I$ be a compact set of the form (2). Then there exist a component P of $J = clos\bigcup_{n=1}^{\infty} J_n$, a non-empty compact set $L \imath M$ with $dist(L, M\setminus L) > 0$, and a map $g \in M(\mathcal{E})$ such that $g|P$ is the identity map and $g(L)$ is the midpoint of P. Consequently, $M \in \mathcal{E}$.*

PROOF. We are going to prove the lemma by transfinite induction on the depth of the set J. Clearly, $d(J) \geq 1$.

Let $d(J) = 1$. If f has only one limit component, then it suffices to use Lemma 11. So, let $\mathrm{Ker}(J) = \{P_1, \ldots, P_r\}$ for some positive integer $r > 1$. Consider a decomposition $M = M_1 \cup \ldots \cup M_r$ where M_i, $i = 1, 2, \ldots, r$, are mutually disjoint compact sets with $P_i \imath M_i$. For every $i = 1, 2, \ldots, r$, the set M_i satisfies the hypothesis of Lemma 11, and thus there are a non-empty compact set $L_i \imath M_i$ with dist $(L_i, M_i\setminus L_i) > 0$ and a map $g_i \in M_i(\mathcal{E})$ such that $g_i|P_i = id$ and $g_i(L_i) = m_i$ where m_i is the midpoint of P_i. Now take

$$g = \bigcup_{i=1}^{r}(g_i|M_i\setminus L_i \cup \langle L_i \rightarrow M_{i+1(\mathrm{mod}\ r)}\rangle)$$

and put $P = P_1$ and $L = L_r$. Clearly, g, P and L have all the desired properties and thus $M \in \mathcal{E}$.

Now suppose that the lemma holds for every set M of the form (2) such that the depth of the corresponding set J is less than $\alpha > 1$ and take a set M with $d(J) = \alpha$. We are going to prove that the lemma holds for this set M. We may assume that $\mathrm{Ker}(J)$ contains only one component P of J, since in the opposite case one can use the same argument as above, when $d(J) = 1$. Further, for the same reasons as in the proof of Lemma 11, we may assume that $P < M\setminus P$. Since $d(P|J) = d(J) = \alpha > 1$, there are mutually disjoint compact sets M_k, $k = 1, 2, \ldots$ such that $M = P \cup \bigcup_{k=1}^{\infty} M_k$, $P < \ldots < M_k < \ldots < M_2 < M_1$ and each of the sets M_k contains infinitely many intervals. Hence, for every $k = 1, 2, \ldots$, the set M_k is of the form (2), i.e., $M_k = \bigcup_{n=1}^{\infty} J_n^k \cup C^k$ where all the sets C^k and J_n^k, $n = 1, 2, \ldots$, are mutually disjoint. Here J_n^k, $n = 1, 2, \ldots$, are compact intervals and C^k is a countable set. Denote $J_k = \mathrm{clos}(\bigcup_{n=1}^{\infty} J_n^k)$. Since $\mathrm{Ker}\ J = \{p\}$ and $d(J) = \alpha$, we have $d(J_k) < \alpha$ for every k. By the induction hypothesis, the lemma holds for every M_k. Thus, for every k there are a component P_k

of J_k, a non-empty compact set $L_k \subset M_k$ with dist $(L_k, M \backslash L_k) > 0$, and a map $g_k \in M_k(\mathcal{E})$ such that $g_k | P_k = id$ and $g_k(L_k) = m_k$ where m_k is the midpoint of P_k. Now take

$$g = id_P \cup \bigcup_{k=1}^{\infty} g_k | M_k \backslash L_k \cup \bigcup_{k=2}^{\infty} \langle L_k \to m_{k-1} \rangle \cup \langle L_1 \to m \rangle$$

where m is the midpoint of P. Finally, put $L = L_1$. Then g, P and L have all the desired properties, and thus $M \in \mathcal{E}$.

The proof of the lemma is finished. □

5. Proofs of Main Results

Proof of Theorem 1. (i) $\mapsto$ (ii). Let (i) be fulfilled, $F(u,v) = (f(u), g_u(v))$. Then M is a non-empty closed subset of I and the set $\{\alpha\} \times M$ is strongly F-invariant. So, $f(\alpha) = \alpha$ and $g(M) = M$ where $g = g_\alpha$. Suppose M is of the form (1). Clearly, C is nowhere dense. Since C is countable and $g(M) = M$, the intervals J_i are permuted by g, i.e., they form one or several cycles of intervals. Call an interval J_i isolated or limit if its distance from C is positive or zero, respectively. Since M is assumed to be of the form (1), there is at least one isolated interval. Denote by $\mathcal{A}$ the union of all isolated intervals. Clearly, dist$(\mathcal{A}, M \backslash \mathcal{A}) > 0$. The set $\mathcal{A}$ cannot be g-invariant, since otherwise the set $\{\alpha\} \times \mathcal{A}$ would be F-invariant and by Lemma 3, the set $\{\alpha\} \times M$ would not be an ω-limit set of F.

Thus there is an isolated interval $\mathcal{K}_1$ such that the interval $\mathcal{K}_2 = g(\mathcal{K}_1)$ is limit. Consider the g-cycle of intervals $\mathcal{K}_1 \mapsto \mathcal{K}_2 \mapsto \ldots \mapsto \mathcal{K}_r \mapsto \mathcal{K}_1$ generated by $\mathcal{K}_1$. Using the continuity of g and the nowhere density of C one can find mutually disjoint neighborhoods U_i of $\mathcal{K}_i$, $i = 1, 2, \ldots, r$ such that if we denote $Q_i = U_i \cap M$, then $Q_j = \mathcal{K}_j$ whenever $\mathcal{K}_j$ is isolated, $g(Q_i) \subset Q_{i+1} (\text{mod } r)$ and dist$(Q_i, M \backslash Q_i) > 0$ for $i = 1, 2, \ldots, r$. Now denote $\bigcup_{i=1}^{r} Q_i$ by Q and suppose that $Q = M$. Then, since $Q_1 = \mathcal{K}_1$, no point from $Q = M$ is mapped by g into the non-empty set $Q_2 \backslash \mathcal{K}_2$. This contradicts the fact that $g(M) = M$.

So $M \backslash Q \neq \emptyset$. Then dist$(Q, M \backslash Q) > 0$ and $F(\{\alpha\} \times Q) \subset \{\alpha\} \times Q$ and so by Lemma 3, the set $\{\alpha\} \times M$ cannot be an ω-limit set of F. This contradiction finishes the proof of (i) $\mapsto$ (ii).

(ii) $\mapsto$ (i). Owing to Lemma 2 it suffices to prove that (ii) implies that $M \in \mathcal{E}$. So let (ii) be fulfilled, i.e., let $M \subset I$ be a non-empty compact set which is not of the form (1). First of all realize that if M is nowhere dense or a union of finite number of intervals, then we are done since by [ABCP,BS] such sets are ω-limit for maps from $C(I,I)$. Further, if M has uncountably many components, then by Lemma 7, $M \in \mathcal{E}$ and we are done again. Finally, if M is a compact subset of I of the form (2), then $M \in \mathcal{E}$ by Lemma 12.

So it remains to consider the case when M is a compact subset of I of the form $M = J_1 \cup J_2 \cup \ldots \cup J_n \cup C$ where n is a positive integer, J_i, $i = 1, 2, \ldots, n$, are closed intervals, C is a non-empty countable set, all the sets J_i and C are mutually disjoint, and $\text{dist}(C, J_i) = 0$ for every $i = 1, 2, \ldots, n$. Then take mutually disjoint compact intervals V_i, $i = 1, 2, \ldots, n$ such that for every i, V_i is a neighborhood of J_i and $\bigcup_{i=1}^n V_i \supset M$. Denote $C_i = C \cap V_i$ and $\mathcal{K}_i = J_i \cup C_i$. According to Lemma 10, for every $i = 1, 2, \ldots, n$ there is a map $g_i \in \mathcal{K}_i(\mathcal{E})$ and a last portion L_i of $\mathcal{K}_i$ such that $g_i | J_i = id$ and $g_i(L_i) = m_i$ where m_i is the midpoint of J_i. Then $f = \bigcup_{i=1}^n (g_i | \mathcal{K}_i \setminus L_i \cup \langle L_i \to m_{i+1 (\text{mod } n)} \rangle)$ belongs to $M(\mathcal{E})$ and thus $M \in \mathcal{E}$.

Proof of Theorem 2. Without loss of generality we can assume that $\alpha = 0$. Owing to Theorem 1, it suffices to consider the case when M is of the form (1). Let M_2 be the union of those intervals on the right hand side of (1) which have positive distances from C, and let $M_1 = M \setminus M_2$. Then both the sets M_1 and M_2 are non-empty, and $\text{dist}(M_1, M_2) > 0$.

Fix $m_1 \in M_1$. From the (ii) $\mapsto$ (i) part of the proof of Theorem 1 we get that $M_1 \in \mathcal{E}$. Similarly as in the proof of Lemma 2, there is an $f_1 \in C(M_1, M_1)$ such that for every $\varepsilon > 0$ there is an ε-chain of f_1 which is an ε-net for M_1 and starts at m_1. Take a sequence ε_i, $i = 1, 2, \ldots,$ $\varepsilon_i \searrow 0$ and a corresponding sequence c_i of such chains. Denote $c_i = \{m_1 = y_1^i, y_2^i, \ldots, y_{k(i)}^i\}$, $i = 1, 2, \ldots$. Clearly, we can assume that $k(1) < k(2) < \ldots < k(i) < \ldots$, and that for every i, the chain c_i is the concatenation of at least two copies of a chain.

Further, it is well known (see, e.g., [ABCP]) that there is an $f_2 \in C(M_2, M_2)$ such that for some $m_2 \in M_2$, the set $\text{orb}_{f_2}(m_2)$ is dense in M_2.

Let $a_r^i = 2^{1-r} - 2^{-i-r}$ for $r = 1, 2, \ldots, k(i)$ and $i = 1, 2, \ldots$. For every $r = 1, 2, \ldots$ there is a positive integer j such that a_r^j is defined. Note that $2^{-r} < a_r^j < a_r^{j+1} < \ldots < 2^{1-r}$ and $\lim_{n \to \infty} a_r^{j+n} = 2^{1-r}$. Define points $\mathcal{A}_r^i = [a_r^i, y_{k(i)+1-r}^i]$ and $\mathcal{B}_r^i = [a_r^i, f_2^{k(i)-r}(m_2)]$ for $r = 1, 2, \ldots, k(i)$ and $i = 1, 2, \ldots$. Denote

$$
\begin{aligned}
\mathcal{K} &= ((\{0\} \cup \{2^{-n},\ n = 0, 1, 2, \ldots\}) \times M) \\
&\cup \bigcup_{i\ odd} \bigcup_{r=1}^{k(i)} \{\mathcal{A}_r^i\} \cup \bigcup_{i\ even} \bigcup_{r=1}^{k(i)} \{\mathcal{B}_r^i\}.
\end{aligned}
$$

Then $\mathcal{K}$ is a compact subset of I^2. We are going to define a map $\varphi \in C_\Delta(\mathcal{K}, I^2)$. For any points $z_1 \in M_1$ and $z_2 \in M_2$ put

$$\varphi([0, z_t]) = [0, f_t(z_t)],\ t = 1, 2;$$

$$\varphi([1, z_t]) = [0, m_{t(mod 2)+1}],\ t = 1, 2;$$

$$\varphi([2^{-n}, z_t]) = [2^{1-n}, f_t(z_t)],\ t = 1,2 \text{ and } n = 1,2,\ldots;$$
$$\varphi(\mathcal{A}^i_s) = \mathcal{A}^i_{s-1},\ s = 2,3,\ldots k(i) \text{ and } i = 1,3,5,\ldots;$$
$$\varphi(\mathcal{A}^i_1) = \mathcal{B}^{i+1}_{k(i+1)},\ i = 1,3,5,\ldots;$$
$$\varphi(\mathcal{B}^i_s) = \mathcal{B}^i_{s-1},\ s = 2,3,\ldots,k(i) \text{ and } i = 2,4,6,\ldots;$$
$$\varphi(\mathcal{B}^i_1) = \mathcal{A}^{i+1}_{k(i+1)},\ i = 2,4,6,\ldots.$$

Then φ is a map from $C_\Delta(\mathcal{K}, I^2)$, and thus, by Lemma 1, it has an extension $\Phi \in C_\Delta(I^2, I^2)$. It is not difficult to see that $\omega_\Phi(\mathcal{A}^1_{k(1)}) \cap I_0 = \{0\} \times M$, which finishes the proof. □

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