Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 35–35

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Questions Concerning the Inverse Function Theorem

There are many inverse function theorems (and related open mapping theorems), each having several equivalent formulations. The one we are concerned with can be stated as follows.

Theorem (IFT). *Suppose that U is a region (non-empty, open, connected set) in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$, f is continuously differentiable on U, and $Jf(x)$, the Jacobian determinant of f at x, is nonzero for all $x \in U$. Then for each $a \in U$ there exists an open set V of $\mathbb{R}^n$ such that $a \in V \subset U$, $W := f(V)$ is open and the restriction of f to V is a bijection whose inverse is continuously differentiable on W.*

The question in question is this: Is IFT true if the word "continuously" is deleted (twice)?

This problem was posed about a decade ago by M. Laczkovich and myself (independently).

In the case $n = 1$ it follows from the intermediate value property of derivatives that the answer to our question is yes.

If $n \geq 2$ then, with the aid of Brouwer degree theory, it is possible to prove that f is an open map. Thus , in order to obtain an affirmative answer to our question, it would suffice to show that f is locally one–to–one.

As far as I know, it is not even known if, for such an f, the Jacobian determinant of f must either be everywhere positive or everywhere negative.

If the answer to our question is yes, one might ask about replacing the differentiability assumption by the mere existence of the relevant partial derivatives.