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Totalization: Newton's Problem and Fourier's Problem

We are concerned with the following two problems:

Problem 1 (Newton's Problem) *Given a derivative, how can we recover its primitive?*

Problem 2 (Fourier's Problem) *Given the limit of an everywhere convergent trigonometric series, how can we recover the series?* The two problems were named by Denjoy, who solved both and considered the solutions to be his two greatest accomplishments. Simpler solutions were discovered by others, and these were harshly criticized by Denjoy for not being "constructive."

The two problems are closely related. Because of theorems of Riemann and Rajchman–Zygmund, respectively, the second problem can be solved by solving either of the following:

Problem 2a *Given the Schwarz derivative of a continuous function, how can we recover the function?*

Problem 2b *Given the approximate symmetric derivative of a measurable function, how can we recover (almost everywhere) the function?*

There is also a natural intermediate problem:

Problem $1\frac{1}{2}$ *Given the symmetric derivative of a continuous function, how can we recover the primitive?*

Denjoy's solution to Problem 1 is well known and not too difficult. He calls the method "totalization." His solution to Problem $1\frac{1}{2}$ is much more complicated and involves a finer analysis of perfect sets. Doubts have been raised as to whether this "symmetric totalization" is really constructive. His solution to Problem 2 follows the path 2a and is extremely difficult and confusing.

The goal is to provide new solutions to these problems based upon the "Henstock–Kurzweil" or "Riemann-complete" method. In order to avoid criticism of being "non-constructive" we must first decide what this means. We therefore examine Denjoy's original totalization procedure as it applies to Problem 1. It immediately becomes apparent how the same process works for inverting approximate derivatives and why it runs into trouble for symmetric derivatives. It also becomes clear in what sense the derivative is "given" (see below). When we compare "totalization" to the following easy solution of Newton's problem: First, find an appropriate $F(x)$ and then show that $F'(x) = f(x)$. This "trivial" solution is supposed to represent the ultimate folly of the non-constructive approaches. Nevertheless, we argue that any definition including this one represents some sort of finding the defined object. Therefore, it is not really "constructive" vs. "non-constructive" but rather "how constructive can we be?" We carry out a hypothetical argument between "totalization" and the "trivial solution" and illustrate that it is not immediately clear that one has any "constructive" advantage over the other. The argument eventually takes us to Recursive Descriptive Set Theory, where we are finally able to give a precise definition for what Denjoy meant by "constructive": CONSTRUCTIVE MEANS HYPERARITHMETICAL IN THE BOREL CODES. We explain the definition and support it with the following compelling evidence: A theorem of Ajtai says that, according to this definition, "totalization" is "constructive". A theorem of Dougherty and Kechris says that no integration procedure could ever be "Borel in the Borel codes". When we combine this with the fact that having a Borel code for a derivative is computationally equivalent to having a method of determining its extreme values on perfect sets, that such a procedure was taken for granted in Denjoy's original process, and that some similar procedure must be taken for granted in any countable integration process, we see that no method of integration could be "more constructive" in a substantial way. Finally, we mention that because of work which is essentially due to Kleene, this definition really does capture Denjoy's original intention; it even provides an actual computer program which calculates the primitive in a transfinite but countable sequence of steps.

We then proceed to work out a detailed "constructive" solution to Problem $1\frac{1}{2}$. Briefly, the technique takes the Riemann–complete definition for a symmetric integral and using elementary recursion theory and partitioning arguments, transforms it into an equivalent hyperarithmetical definition. This definition accordingly represents a symmetric integration process which is "constructive" in the above precise way.

The real goal of all of this is to use the same techniques to solve Problem 2. Unlike Denjoy's original solution, there would presumably be no doubt as to the "constructive" nature of such a solution. Details have not yet been worked out. However, it is conjectured that the approach outlined for the symmetric derivative, will be able to turn any Riemann type integral into a constructive one. Riemann–type integrals are known for both 2a and 2b.

For references and more details see the accompanying INROADS article.