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Generalized Lebesgue Points *

Let f be Lebesgue integrable on $[-\pi, \pi]$ and let $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt+\sin nt)$, be the Fourier series of f. A well known result of Lebesgue, states that if x is a Lebesgue point of f, then

$$\lim_{n\to\infty} \sigma_n(x) = f(x), \qquad (*)$$

where we adopt the usual notation $\sigma_n(x) = \frac{s_0(x)+s_1(x)\cdots+s_{n-1}(x)}{n}$, and $s_0(x) = \frac{a_0}{2}, s_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt)$, $m = 1, 2\ldots$. In the paper being summarized here we prove that if f is Denjoy-Perron integrable on $[-\pi, \pi]$, and if x is a generalized Lebesgue point of f, then (*) holds. Since it is shown that a Lebesgue point of a Lebesgue integrable function is a generalized Lebesgue point this result gives a generalization of the classical result. Prior to giving a precise statement of the main result, we list some background results.

A well-known result of Féjer states that if f is continuous at x, then its Fourier series is $(C, 1)$-summable at x to $f(x)$. Over the years this continuity condition has been relaxed. Lebesgue showed that if f is Lebesgue integrable, then its Fourier series is $(C, 1)$-summable to $f(x)$ at any point where $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h}|f(x) - f(t)|dt = 0$, i.e. at the full measure set of **Lebesgue points** of f. Fatou showed that if all we are concerned with is A-summability, then $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h}\big(f(x) - f(t)\big)dt = 0$, will do. These are called points of **C-continuity**. What happens if the function is only D^*-integrable? There is a theorem of Marcinkiewicz quoted in Čelidze & Džvaršeĭšvili as: If a function is D-integrable, then it is $(C, 1)$-summable at all points of C-continuity. This seems to be a misquote since then we could extend Fatou's result to $(C, 1)$-summability. There is an example in Zygmund of a function whose differentiated series is not $(C, 1)$-summable at a point where it has a derivative; however it is not clear that this function is ACG^*, or even ACG. A reading of the original paper of Marcinkiewicz seems to suggest the correct result is: A D-integrable function is $(C, 1)$-summable at almost all points of C-continuity. In the case of Lebesgue integrable functions this implies a weaker form of Lebesgue's result.

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Our aim is to define a class of points at which the Fourier series of a D^*-integrable function is $(C,1)$-summable. First we must give a criterion for D^*-integrability, due to the present author. It is set in the Henstock-Kurzweil theory.

Theorem 1 *$f \in D^*([a,b])$ iff there exist a decomposition of $[a,b]$ into a sequence of disjoint sets $E_n, n \in \mathbb{N}$ with the following properties:*

(a) for all $n \in \mathbb{N}$ the finite union $\cup_{k\le n} E_k$ is closed;

(b) for all $n \in \mathbb{N}$, $f \in \mathcal{L}(E_n)$;

(c) if for all $n \in \mathbb{N}$, F_n denotes the Lebesgue primitive of $f|_{E_n}$, then $\sum_{n\in\mathbb{N}} F_n$ converges uniformly;

(d) for all $n \in \mathbb{N}$ there exists a gauge δ_n such that

$$\forall x \notin \bigcup_{k\le n} E_k,\ (x-\delta_n(x), x+\delta(x)) \subset (a,b)\setminus \bigcup_{k\le n} E_k$$

and if $\tau_n = \sup \sum_{x\notin \cup_{k\le n} E_k} f(x)(v-u)$, the sup being over all δ_n-fine partitions of $[a,b]$, and $x \notin \cup_{k\le n} E_k$ then $\lim_{n\to\infty} \tau_n = 0$.

Then the D^*-primitive of f is $\sum_{n\in\mathbb{N}} F_n$. With the notation of this theorem we call x a **generalized Lebesgue point** if the following limit exists uniformly, (in n).

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \left| \sum_{k\le n} (f(t) - f(x))\, |_{E_k}(t) \right| dt = 0.$$

It follows immediately that if f is Lebesgue integrable, then a Lebesgue point is a generalized Lebesgue point; and if f is continuous at x, then x is a generalized Lebesgue point. The main result of the paper is then:

Theorem 2 *If $f \in D^*(-\pi,\pi)$ then the Fourier series of f is $(C,1)$-summable at all generalized Lebesgue points.*

Unfortunately, so far we have only been able to show that if $f \in D^*(-\pi,\pi)$, then the set of generalized Lebesgue points is a dense set of positive measure.

References

[1] V.G.Čelidze and A.G.Džvaršeĭšvili, Teoriya Integrala Denjoy i Nekotorye eë Proloženiya, *Tbilisi. Gos. Univ., Tbilisi*; English translation.

[2] Liu Gen-qian, *On necessary conditions for Henstock integrability*, submitted.